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# ARCWISE ANALYTIC STRATIFICATION, WHITNEY FIBERING CONJECTURE AND ZARISKI EQUISINGULARITY

ADAM PARUSIŃSKI AND LAURENTIU PAUNESCU

**ABSTRACT.** In this paper we show Whitney’s fibering conjecture in the real and complex, local analytic and global algebraic cases.

For a given germ of complex or real analytic set, we show the existence of a stratification satisfying a strong (real arc-analytic with respect to all variables and analytic with respect to the parameter space) trivialization property along each stratum. We call such a trivialization arc-wise analytic and we show that it can be constructed under the classical Zariski algebro-geometric equisingularity assumptions. Using a slightly stronger version of Zariski equisingularity, we show the existence of Whitney’s stratified fibration, satisfying the conditions (b) of Whitney and (w) of Verdier. Our construction is based on Puiseux with parameter theorem and a generalization of Whitney interpolation. For algebraic sets our construction gives a global stratification.

We also give several applications of arc-wise analytic trivialization, mainly to the stratification theory and the equisingularity of analytic set and function germs. In the real algebraic case, for an algebraic family of projective varieties, we show that Zariski equisingularity implies local triviality of the weight filtration.

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## Introduction and statement of results.

In 1965 Whitney stated the following conjecture.

**Conjecture.** [Whitney fibering conjecture, [62] section 9, p.230] *Any analytic subvariety  $V \subset U$  ( $U$  open in  $\mathbb{C}^n$ ) has a stratification such that each point  $p_0 \in V$  has a neighborhood  $U_0$  with a semi-analytic fibration.*

By a semi-analytic fibration Whitney meant the following (that has nothing to do with the notion of semi-analytic set introduced about the same time by Łojasiewicz in [33]). Let  $p_0$  belong to a stratum  $M$  and let  $M_0 = M \cap U_0$ . Let  $N$  be the analytic plane orthogonal to  $M$  at  $p_0$  and let  $N_0 = N \cap U_0$ . Then Whitney requires that there exists a homeomorphism

$$\phi(p, q) : M_0 \times N_0 \rightarrow U_0,$$

complex analytic in  $p$ , such that  $\phi(p, p_0) = p$  ( $p \in M_0$ ) and  $\phi(p_0, q) = q$  ( $q \in N_0$ ), and preserving the strata. He also assumes that for each  $q \in N_0$  fixed,  $\phi(\cdot, q) : M_0 \rightarrow U_0$  is a complex analytic embedding onto an analytic submanifold  $L(q)$  called the fiber (or the leaf) at  $q$ , and thus  $U_0$  fibers continuously into submanifolds complex analytically diffeomorphic to  $M_0$ . Note that due to the existence of continuous moduli it is in general impossible to find  $\phi(p, q)$  complex analytic in both variables, see [62].

Whitney stated his conjecture in the context of the conditions (a) and (b) for stratifications that he introduced in [61]. These regularity conditions imply the topological triviality (equisingularity) along each stratum; this trivialization is obtained by the flow of "controlled" vector fields and does not imply the existence of a fibration as required in Whitney's conjecture. Thus Whitney conjectured the existence of a better trivialization, given by his fibration, that should moreover imply the regularity conditions. As Whitney claims in [62] a semi-analytic (in his sense) fibration ensures the continuity of the tangent spaces to the leaves of the fibration and hence Whitney's (a) condition for the stratification. This seems to be not so obvious. We recall Whitney's argument in Subsection 7.4, but to complete it we need an extra assumption. To have the condition (b), quoting Whitney, "one should probably require more than just the continuity of  $\phi$  in the second variable".

Whitney's fibering conjecture as stated above was proven in local analytic and global projective cases in Theorem 6.1 of [18]. But it is not clear to us whether  $\phi$  of [18] assures the continuity of the tangent spaces to the leaves or the condition (b). In the real algebraic case an analogue of Whitney's conjecture was proven in [17]. In this case the continuity of the tangent spaces is not clear either.

Whitney's fibering conjecture has been studied in the context of abstract  $C^\infty$  stratified spaces and topological equisingularity, cf. [41], [42], [43]. Assuming that the conjecture holds, Murolo and Trotman show in [43], in particular, a horizontally- $C^1$  version of Thom's first isotopy theorem.

**0.1. Ehresmann Theorem.** Whitney's conjecture is consistent with the following holomorphic version of the Ehresmann fibration theorem, see [59]. Let  $\pi : \mathcal{X} \rightarrow B$  be a proper

holomorphic submersion of complex analytic manifolds. Then, for every  $b_0 \in B$  there is a neighborhood  $B_0$  of  $b_0$  in  $B$  and a  $C^\infty$  trivialization

$$\phi(p, q) : B_0 \times X_0 \rightarrow \mathcal{X}_{B_0},$$

holomorphic in  $p$ , where  $X_0 = \pi^{-1}(b_0)$ ,  $\mathcal{X}_0 = \pi^{-1}(B_0)$ . Note that  $\phi$  can be made real analytic but, in general, due to the presence of continuous moduli, not holomorphic. This version of Ehresmann's theorem is convenient to study the variation of Hodge structures in families of Kähler manifolds, see [59].

As we show in this paper there are no continuous moduli for complex analytic families of singular complex analytic germs (the fibres can be global for families of algebraic varieties) provided  $\varphi$  is assumed complex analytic in  $p$  and real arc-analytic in  $q$ , see Theorems 7.6 and 9.1 and Lemma 7.4.

**0.2. Statement of main results.** In this paper we show Whitney fibering conjecture in the real and complex, local analytic and global algebraic cases. For this, for a given germ of complex or real analytic set, we show the existence of a stratification that can locally be trivialized by a map  $\phi(p, q)$  that is not only analytic in  $p$  and continuous in both variables but also arc-wise analytic, see Definition 1.2. In particular, both  $\phi$  and  $\phi^{-1}$  are analytic on real analytic arcs. Moreover, for every real analytic arc  $q(s)$  in  $N_0$ ,  $\phi(p, q(s))$  is analytic in  $p$  and  $s$ . As we show in Proposition 1.3 this guaranties the continuity of tangent spaces to the fibres and hence Whitney's (a)-condition on the stratification (both in the real and complex cases). Then, by additionally requiring that the trivialization preserves the size of the distance to the stratum, we show the existence of Whitney's fibration satisfying conditions (b) of Whitney and (w) of Verdier [57]. We call such an arc-wise analytic trivialization regular along the stratum, Definition 1.5.

**Theorem** (Theorem 7.6). *Let  $\mathcal{X} = \{X_i\}$  be a finite family of analytic subsets of an open  $U \subset \mathbb{K}^N$ . Let  $p_0 \in U$ . Then there exist an open neighborhood  $U'$  of  $p_0$  and an analytic stratification of  $U'$  compatible with each  $U' \cap X_i$  admitting regular arc-wise analytic trivialization along each stratum.*

In Section 8 we extend these results to stratifications of analytic functions. Recall that a stratification of a  $\mathbb{K}$ -analytic function  $f : X \rightarrow \mathbb{K}$  is a stratification of  $X$  such that the zero set  $V(f)$  of  $f$  is a union of strata. The Theorem 8.2 together with Proposition 1.13 imply the following result.

**Theorem** ( $\mathbb{K} = \mathbb{C}$ ). *If a stratification of  $f$  admits an arc-wise analytic trivialization along a stratum  $S \subset V(f)$  then it satisfies the Thom condition  $(a_f)$  along this stratum. If such trivialization is moreover regular along  $S$ , then it satisfies the strict Thom condition  $(w_f)$  along  $S$ .*

We also give an analogous result in the real case using the notion of regularity of a function for an arc-wise analytic trivialization, defined in Subsection 1.3. Thom's regularity conditions are used to show topological triviality of functions along strata. We discuss it in details in Section 8, where we develop three different constructions guarantying such triviality.

In Section 9 we treat the algebraic case. By reduction to the homogeneous analytic case we show the following results.

**Theorem** (Theorem 7.7). *Let  $\{V_i\}$  be a finite family of algebraic subsets of  $\mathbb{P}_{\mathbb{K}}^n$ . Then there exists an algebraic stratification of  $\mathbb{P}_{\mathbb{K}}^n$  compatible with each  $V_i$  and admitting semialgebraic regular arc-wise analytic trivializations along each stratum.*

**Theorem** (Theorem 9.1). *Let  $T$  be an algebraic variety and let  $\mathcal{X} = \{X_k\}$  be a finite family of algebraic subsets  $T \times \mathbb{P}_{\mathbb{K}}^{n-1}$ . Then there exists an algebraic stratification  $\mathcal{S}$  of  $T$  such that for every stratum  $S$  and for every  $t_0 \in S$  there is a neighborhood  $U$  of  $t_0$  in  $S$  and a semialgebraic arc-wise analytic preserving the family  $\mathcal{X}$  trivialization of  $\pi$*

$$(0.1) \quad \Phi : U \times \mathbb{P}_{\mathbb{K}}^{n-1} \rightarrow \pi^{-1}(U),$$

$\Phi(t_0, x) = (t_0, x)$ , where  $\pi : T \times \mathbb{P}_{\mathbb{K}}^{n-1} \rightarrow T$  denotes the projection.

The arc-wise analytic analytic triviality is particularly friendly to the curve selection lemma argument. Recall that in analytic geometry many properties can be proven by checking them along real analytic arcs. We use this argument often in this paper. We refer the reader to [9], [60], [33], [21] or [38]. To prove the classical regularity conditions, (a) of Whitney, (w) of Verdier, or Thom's conditions (a<sub>f</sub>) or (w<sub>f</sub>), we use a wing lemma type argument originated by Whitney in [61], see Proposition 7.3. Arc-wise analytic trivializations naturally provide such wings. For instance, in Whitney's notation, if  $q(s)$  is a real analytic arc in  $N_0$  then  $\phi(p, q(s))$  constitutes such an arc-wise analytic wing. Moreover, arc-wise analytic trivializations preserve the multiplicities and the singular loci of the sets they trivialize, see Propositions 1.11 and 1.14 for precise statements.

Thus in this paper we redefine and reprove most of the classical results of stratification theory of real and complex, analytic (only locally) and algebraic sets, in order to get local arc-wise analytic trivializations. Our approach is based on the classical Puiseux with parameter theorem and the algebro-geometric equisingularity of Zariski (called also Zariski equisingularity). Our main tool in construction of arc-wise analytic trivializations is Theorem 3.3, that says that Zariski equisingularity implies arc-wise analytic triviality. To show it, we use Whitney interpolation that we adapt to arc-analytic geometry. This is explained in Appendix I.

Besides the proofs of Puiseux with parameter theorem and the curve selection lemma this paper is virtually self-contained. Moreover, our method, since is based on the Zariski equisingularity, is constructive; it involves the computation of the discriminants of subsequent linear projections.

**0.3. Zariski Equisingularity.** Let  $V$  be a real or a complex analytic variety. Then there exists a stratification  $\mathcal{S}$  of  $V$  such that  $V$  is equisingular along each stratum  $S$ . There are several different notions of equisingularity, the basic is the topological one, with many possible refinements, stratified topological triviality for instance. Whitney introduced in [62], [61], the regularity conditions (a) and (b) that guarantee, by the Thom-Mather first isotopy theorem, the topological equisingularity along each stratum. He showed in [61] that any

complex analytic variety admits (a) and (b) regular stratifications. The real analytic case was established in [33] and the subanalytic case in [21].

Besides isotopy lemmas applied to stratified spaces, the topological equisingularity can be obtained by means of Zariski equisingularity, as shown by Varchenko in [54], [55], [56]. Zariski's definition, see [65], is recursive and is based on the geometry of discriminants. Let  $V \subset \mathbb{K}^N$  be a hypersurface. We say that  $V$  is Zariski equisingular along stratum  $S$  at  $p \in S$  if, after a change of a local system of coordinates, the discriminant of a linear projection  $\pi : \mathbb{K}^N \rightarrow \mathbb{K}^{N-1}$  restricted to  $V$  is equisingular along  $\pi(S)$  at  $\pi(p)$ . The kernel of  $\pi$  should be transverse to  $S$  and  $\pi$  restricted to  $V$  should be finite at  $p$ . Stronger notions of equisingularity are obtained if one assumes that the kernel of  $\pi$  is not contained in the tangent cone to  $V$  at  $p$  (transverse Zariski equisingularity) or that  $\pi$  is generic (generic Zariski equisingularity).

The special case, when  $S$  is of codimension one in  $V$ , was studied by Zariski in [63]. Note that in this case  $V$  can be considered as a family of plane curves parameterized by  $S$ . As Zariski shows, in this case Zariski equisingularity is equivalent to Whitney's conditions (a) and (b) on the pair of strata  $V \setminus S, S$ . Such equisingular families admit uniform Puiseux representation parameterized by  $S$ , that in literature is referred to as Puiseux with parameter theorem. We recall it in Subsection 2.1.

In this paper we show that Zariski equisingularity implies arc-wise analytic triviality. In the case of Zariski transverse equisingularity, arc-wise analytic triviality can also be obtained regular.

**Theorem** (see Theorems 3.3 and 4.2). *If a hypersurface  $V \subset \mathbb{K}^N$  is Zariski equisingular along stratum  $S$  at  $p \in S$ , then there is a local arc-wise analytically trivialization of  $\mathbb{K}^N$  along  $S$  at  $p$  that preserves  $V$ .*

Our proof is different from that of Varchenko and is based on Whitney interpolation that gives a precise algorithmic formula for such a trivialization. The main idea is the following. Suppose  $V$  is Zariski equisingular along  $S$  and  $\pi : \mathbb{K}^N \rightarrow \mathbb{K}^{N-1}$  is the projection giving this equisingularity. Thus, by inductive assumption, there is, given by a precise formula, an arc-wise analytic trivialization of  $\pi(V)$  along  $\pi(S)$ . This trivialization is then lifted to a trivialization of  $V$  along  $S$  by our version of Whitney interpolation. Therefore the lift is continuous, subanalytic, and, by Puiseux with parameter theorem, arc-wise analytic. This latter conclusion is obtained thanks to the arc-wise analyticity in the inductive assumption, see Remark 3.4.

For an analytic function germ  $F$  we denote by  $F_{red}$  its reduced (i.e. square free) form. Let  $(Y, y)$  be the germ of a  $\mathbb{K}$ -analytic space. For a monic polynomial  $F \in \mathcal{O}_Y[Z]$  in  $Z$  we often consider the discriminant of  $F_{red}$ . If  $Y$  has arbitrary singularities then this discriminant should be replaced by an appropriate generalized discriminant that is a polynomial in coefficients of  $F$ , see Appendix II.

Finally we note that Zariski equisingularity can be used to trivialize not only hypersurfaces but also analytic spaces of arbitrary embedding codimension. This follows from the fact that if a hypersurface  $V = \cup V_i$  is Zariski equisingular along  $S$  and  $V = \cup V_i$  be the decomposition

of  $V$  into irreducible components then the arc-wise analytic trivialization preserves each  $V_i$  and hence any set theoretic combination of the  $V_i$ 's.

**0.4. Zariski Equisingularity and regularity conditions on stratifications.** In general, both equisingularity methods, Whitney's (a) and (b) conditions and Zariski equisingularity, give different equisingularity conditions. We recall several classical examples in Section 7.5. By Zariski [63], they coincide for a hypersurface  $V$  along a nonsingular subvariety of codimension 1 in  $V$ .

It was shown by Speder in [52] that Zariski equisingularity obtained by taking generic projections implies the regularity conditions (a) and (b) of Whitney. As it follows from our Theorem 4.2 the assumption that the projections are transverse is sufficient. We also show in Proposition 3.9 that Zariski equisingularity (arbitrary projections) implies equimultiplicity.

Whitney's stratification approach is independent of choice of local coordinates and simple to define. But the trivializations obtained by this method are not explicit and difficult to handle. These trivializations are obtained by integration of "controlled" vector fields whose existence can be theoretically established. Stronger regularity conditions, such as (w) of Verdier [57], or Lipschitz of Mostowski [40], [46], lead to easier constructions of such vector fields. But, in general, even if these vector fields can be chosen subanalytic not much can be said about their flows.

Zariski equisingularity method is more explicit and in a way constructive. It uses the actual equations and local coordinate systems. This can be considered either as a drawback or as an advantage. Zariski equisingularity was used, for instance, by Mostowski [39], see also [3], to show that analytic set germs are always homeomorphic to the algebraic ones. As we show in this paper, Zariski equisingular families can be trivialized by arc-wise analytic trivializations that are explicitly constructed. Moreover these trivializations are subanalytic (semialgebraic in the algebraic case) and therefore various methods, such as the curve selection lemma, can be applied to them.

In this paper we apply Zariski equisingularity to construct stratifications via corank one projections. This method was developed by Hardt and Hardt & Sullivan [15, 16, 17], [18].

**0.5. Applications to real algebraic geometry.** The semialgebraic arc-analytic maps are often used in real algebraic geometry. The arc-analytic maps were introduced by Kurdyka in [27]. It was shown by Bierstone and Milman in [2] (see also [48]) that the semialgebraic arc-analytic maps are blow-analytic. The semialgebraic arc-analytic maps and, closely related, semialgebraic arc-symmetric sets were used in [29], [49]. to show that injective self-morphisms of real algebraic varieties are surjective. For more on this development we refer the reader to [30]. Let us also note that recently studied [23], [24], [25], [12] continuous rational maps are, in particular, arc-analytic and semi-algebraic.

The weight filtration on real algebraic varieties, recently introduced [36, 37], is stable by semialgebraic arc-analytic homeomorphisms. By Theorem 9.1 any algebraic family of algebraic sets is generically semialgebraically arc-wise analytic trivial, and therefore we have the following result.



**Theorem** (see Corollary 9.4). *Let  $T$  be a real algebraic variety and let  $X$  be an algebraic subset of  $T \times \mathbb{P}_{\mathbb{K}}^{n-1}$ . Then there exists a finite stratification  $\mathcal{S}$  of  $T$  such that for every stratum  $S$  and for every  $t_0, t_1 \in S$  the fibres  $X_{t_0}$  and  $X_{t_1}$  have the weight spectral sequences and the weight filtrations isomorphic.*

**0.6. Resolution of singularities and blow-analytic equivalence.** The resolution of singularities can also be used to show topological equisingularity, though in this approach the results are partial and there are still many open problems. What is known is that this method works for the families of isolated singularities, cf. Kuo [26]. Moreover this method gives local arc-analytic trivializations. But little is known if the singularities are not isolated, see e.g. [22]. Let us explain the encountered problem on a simple example. Suppose that  $Y \subset V$  is nonsingular and let  $\sigma : \tilde{V} \rightarrow V$  be a resolution of singularities such that  $\sigma^{-1}(Y)$  is a union of the components of exceptional divisors. Fix a local projection  $\pi : V \rightarrow Y$ . The exceptional divisor of  $\sigma$  as a divisor with normal crossings is naturally stratified by the intersections of its components. Let  $Z \subset Y$  be the closure of the union of all critical values of  $\pi \circ \sigma$  restricted to the strata. By Sard's theorem  $\dim Z < \dim Y$ . We say that  $V$  is *equiresoluble along  $Y$*  if  $Y \cap Z = \emptyset$ . Thus  $V$  is equiresoluble along  $Y' = Y \setminus Z$  and  $\pi \circ \sigma$  is locally topologically (and even real analytically) trivial over  $Y'$ . If  $\sigma$  is an isomorphism over  $V \setminus Y$  (family of isolated singularities case) then this trivialization blows down to a topological trivialization of a neighborhood of  $Y$  in  $V$ . But in the non-isolated singularity case there is no clear reason why a trivialization of  $\pi \circ \sigma$  comes from a topological trivialization of a neighborhood of  $Y$  in  $V$ . Thus, in general, we do not know whether equiresolubility implies topological equisingularity.

As before, one may ask how the equiresolution method is related to the other methods of establishing topological equisingularity. A non-trivial result of Villamayor [58], says that the generic Zariski equisingularity of a hypersurface implies a weak version of equiresolution, see loc. cit. for details, but the main problem remains, it does not show the existence of a topological trivialization that lifts to the resolution space.

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**Notation and terminology.** We denote by  $\mathbb{K}$  the field of real  $\mathbb{R}$  or complex  $\mathbb{C}$  numbers and, unless expressly stated otherwise, we work simultaneously in the real and the complex cases. Therefore we rather say complex analytic than holomorphic, so that for the complex or real case we may abbreviate it as  $\mathbb{K}$ -analytic.

By an analytic space we mean an analytic space in the sense of [44], though, because in the analytic case we work only locally, it is sufficient to consider only analytic set germs. For a  $\mathbb{K}$  analytic space  $X$  by  $Sing(X)$  we denote the set of singular points of  $X$  and by  $Reg(X)$  its complement, the set of regular points of  $X$ . For an analytic function germ  $F$  we denote by  $V(F)$  its zero set and by  $F_{red}$  its reduced (i.e. square free) form. By a real analytic arc we mean a real analytic map  $\gamma : I \rightarrow X$ , where  $I = (-1, 1)$  and  $X$  is a real or a complex analytic space.

## Part 1. Arc-wise analytic trivializations.

### 1. DEFINITION AND BASIC PROPERTIES

Let  $Z, Y$  be  $\mathbb{K}$ -analytic spaces. A map  $f : Z \rightarrow Y$  is called *arc-analytic* if  $f \circ \delta$  is analytic for every real analytic arc  $\delta : I \rightarrow Z$ , where  $I = (-1, 1) \subset \mathbb{R}$ . The arc-analytic maps were introduced by Kurdyka in [27] and subsequently have been used intensively in real analytic and algebraic geometry, see [30]. It was shown by Bierstone and Milman in [2] (see also [48] for a different proof) that the arc-analytic maps with subanalytic graphs are continuous and that the arc-analytic maps with semi-algebraic graphs are blow-analytic, i.e. become real-analytic after composing with blowings-up. Therefore the arc-analytic maps are closely related to the blow-analytic trivialization in the sense of Kuo [26].

In this paper we consider arc-analytic trivialization satisfying some additional properties. We define below the notion of arc-wise analytic trivialization, that is not only arc-analytic with arc-analytic inverse, but it is also  $\mathbb{K}$ -analytic with respect to the parameter  $t \in T$ . For simplicity we always assume that the parameter space  $T$  is nonsingular.

**Definition 1.1.** Let  $T, Y, Z$  be  $\mathbb{K}$ -analytic spaces,  $T$  nonsingular. We say that a map  $f(t, z) : T \times Z \rightarrow Y$  is *arc-wise analytic in  $t$*  if it is  $\mathbb{K}$ -analytic in  $t$  and arc-analytic in  $z$ , that is if for every real analytic arc  $z(s) : I \rightarrow Z$  there is a neighborhood  $U$  of  $T \times \{0\}$  in  $T \times I$  such that the map

$$U \ni (t, s) \rightarrow f(t, z(s))$$

is  $\mathbb{K}$ -analytic in  $t$  and real analytic in  $s$ .

All arc-wise analytic maps considered in this paper are subanalytic and hence continuous.

We stress that even for complex analytic spaces we define the notion of arc-analyticity using only real analytic arcs. (A map of complex analytic spaces  $f : Z \rightarrow Y$ , with  $Z$  nonsingular, that is complex analytic on complex analytic arcs is, by Hartogs Theorem, complex analytic.)

**Definition 1.2.** Let  $Y, Z$  be  $\mathbb{K}$ -analytic spaces and let  $T$  be a nonsingular  $\mathbb{K}$ -analytic space. Let  $\pi : Y \rightarrow T$  be a  $\mathbb{K}$ -analytic map. We say

$$\Phi(t, z) : T \times Z \rightarrow Y$$

is an *arc-wise analytic trivialization of  $\pi$*  if it satisfies the following properties

- (1)  $\Phi$  is a subanalytic homeomorphism,
- (2)  $\Phi$  is arc-wise analytic in  $t$  (in particular it is  $\mathbb{K}$ -analytic with respect to  $t$ ),
- (3)  $\pi \circ \Phi(t, z) = t$  for every  $(t, z) \in T \times Z$ ,
- (4) the inverse of  $\Phi$  is arc-analytic,
- (5) there exist  $\mathbb{K}$ -analytic stratifications  $\{Z_i\}$  of  $Z$  and  $\{Y_i\}$  of  $Y$ , such that for each  $i$ ,  $Y_i = \Phi(T \times Z_i)$  and  $\Phi|_{T \times Z_i} : T \times Z_i \rightarrow Y_i$  is a real analytic diffeomorphism.

Sometimes we say for short that such  $\Phi$  is an arc-wise analytic trivialization if it is obvious from the context what the projection  $\pi$  is.

In the algebraic case we require  $\Phi$  to be semialgebraic and that the stratification is algebraic in the sense we explain in Section 7.

If  $\Phi(t, z) : T \times Z \rightarrow Y$  is an *arc-wise analytic trivialization* then, for each  $z \in Z$ , the map  $T \ni t \rightarrow \Phi(t, z) \in Y$  is a  $\mathbb{K}$ -analytic embedding. We denote by  $L_z$  its image and we call it a *leaf* or a *fiber* of  $\Phi$ . We say that  $\Phi$  *preserves*  $X \subset Y$  if  $X$  is a union of leaves. We denote by  $T_y = T_y L_z$ ,  $y = \Phi(t, z)$ , the tangent space to the leaf through  $y$ .

**1.1. Computation in local coordinates.** Let  $(t_0, z_0) \in T \times Z$ ,  $y_0 = \Phi(t_0, z_0)$ . Choosing local coordinates we may always assume that  $(T, t_0) = (\mathbb{K}^m, 0)$ ,  $(Z, z_0)$  is an analytic subspace of  $(\mathbb{K}^n, 0)$ , and  $(Y, y_0)$  is an analytic subspace of  $(T \times \mathbb{K}^n, 0)$  with  $\pi(t, x) = t$ . Thus we may write

$$(1.1) \quad \Phi(t, z) = (t, \Psi(t, z)).$$

We also suppose that  $L_0 = \Phi(T \times \{0\}) = T \times \{0\}$  as germs at the origin..

Using local coordinates we identify  $T_y$  with an  $m$ -dimensional vector subspace of  $\mathbb{K}^m \times \mathbb{K}^n$  and consider  $T_y$  as a point in the Grassmanian  $G(m, m+n)$ . These tangent spaces are spanned by the vector fields  $v_i$  on  $Y$  defined by

$$(1.2) \quad v_i(\Phi(t, z)) := (\partial/\partial t_i, \partial\Psi/\partial t_i) \quad i = 1, \dots, m.$$

**Proposition 1.3.** *Let  $\Phi(t, z) : T \times Z \rightarrow Y$  be an arc-wise analytic trivialization. Then the vector fields  $v_i$  and the tangent space map  $y \rightarrow T_y$  are subanalytic, arc-analytic, and continuous.*

*Proof.* The subanalyticity follows by the classical argument of subanalyticity of the derivative of a subanalytic map, see [28] Théorème 2.4. Let  $(t(s), z(s)) : (I, 0) \rightarrow (T \times Z, (t_0, z_0))$  be a real analytic arc germ. Consider the map  $\Psi : T \times I \rightarrow \mathbb{K}^n$

$$(1.3) \quad \Psi(t, z(s)) = \sum_{k \geq k_0} D_k(t) s^k.$$

The arc-analyticity of  $v_i$  on  $(t(s), z(s))$  follows from the analyticity of  $(t, s) \rightarrow \partial\Psi(t, z(s))/\partial t_i$ . Finally, subanalytic and arc-analytic maps are continuous, cf. [2] Lemma 6.8.  $\square$

*Remark 1.4.* For  $y = \Phi(t, z)$  fixed,  $\tau \rightarrow \Phi(t + \tau e_i, z)$  is an integral curve of  $v_i$  through  $y$ . Moreover, such an integral curve is unique as follows from (5) of Definition 1.2.

**1.2. Arc-wise analytic trivializations regular along a fiber.** We now define regular arc-wise analytic trivializations along a fiber that will be important for applications in stratification theory including our proof of Whitney fibering conjecture, c.f. section 7. Regular arc-wise analytic trivializations preserve the size of the distance to a fixed fiber.

**Definition 1.5.** We say that an arc-wise analytic trivialization  $\Phi(t, z) : T \times Z \rightarrow Y$  is *regular at*  $(t_0, z_0) \in T \times Z$  if there is a neighborhood  $U$  of  $(t_0, z_0)$  and a constant  $C > 0$  such that for all  $(t, z) \in U$  (in local coordinates at  $(t_0, z_0)$  and  $y_0 = \Phi(t_0, z_0)$ )

$$(1.4) \quad C^{-1} \|\Psi(t_0, z)\| \leq \|\Psi(t, z)\| \leq C \|\Psi(t_0, z)\|,$$

where as in (1.1),  $\Phi(t, z) = (t, \Psi(t, z))$ . We say that  $\Phi$  is *regular along*  $L_{z_0}$  if it is regular at every  $(t, z_0)$ ,  $t \in T$ .

We have the following criterion of regularity, as special case of Proposition 1.7.

**Proposition 1.6.** *The arc-wise analytic trivialization  $\Phi(t, z)$  is regular at  $(0, 0)$  if and only if for every real analytic arc germ  $z(s) : (I, 0) \rightarrow (Z, 0)$ , the leading coefficient of (1.3) is nonzero:  $D_{k_0}(0) \neq 0$ .*

*Moreover, if  $\Phi(t, z)$  is regular at  $(0, 0)$ , then in a neighborhood of  $(0, 0) \in T \times Z$*

$$(1.5) \quad \left\| \frac{\partial \Psi}{\partial t}(t, z) \right\| \leq C \|\Psi(t, z)\|.$$

**1.3. Functions and maps regular along a fiber.** In this section we generalize the notion of regularity for arc-wise analytic trivializations to analytic maps.

**Proposition 1.7.** *Let  $\Phi(t, z) : T \times Z \rightarrow Y$  be an arc-wise analytic trivialization and let  $f : (Y, y_0) \rightarrow (\mathbb{R}^k, 0)$ ,  $y_0 = \Phi(t_0, z_0)$ , be a real analytic map germ. Then the following conditions are equivalent:*

(i) *for a constant  $C > 0$  and for all  $(t, z)$  sufficiently close to  $(t_0, z_0)$*

$$(1.6) \quad C^{-1} \|f(\Phi(t_0, z))\| \leq \|f(\Phi(t, z))\| \leq C \|f(\Phi(t_0, z))\|.$$

(ii) *for every real analytic arc germ  $z(s) : (I, 0) \rightarrow (Z, z_0)$  the leading coefficient  $D_{k_0}$  of*

$$(1.7) \quad f(\Phi(t, z(s))) = \sum_{k \geq k_0} D_k(t) s^k$$

*satisfies  $D_{k_0}(t_0) \neq 0$ .*

(iii) *there is  $C > 0$  such that for all  $(t, z)$  sufficiently close to  $(t_0, z_0)$*

$$(1.8) \quad \left\| \frac{\partial(f \circ \Phi)}{\partial t}(t, z) \right\| \leq C \|f \circ \Phi(t, z)\|.$$

*Proof.* The equivalence of (i) and (ii) follows from the curve selection lemma. Similarly, it is sufficient to show (iii) on every real analytic arc and this follows immediately from (ii). Finally (i) follows from (iii).  $\square$

*Remark 1.8.* If  $\Phi$  is a complex trivialization, then we may understand the left-hand side of (1.8) as the absolute value of the real gradient, or equivalently if  $f$  is complex analytic, as the absolute value of the complex gradient.

**Definition 1.9.** Let  $\Phi(t, z) : T \times Z \rightarrow Y$  be an arc-wise analytic trivialization in  $t$ . We say that an analytic function germ  $f : (Y, y_0) \rightarrow (\mathbb{K}, 0)$ ,  $y_0 = \Phi(t_0, z_0)$ , is *regular for  $\Phi$*  if it satisfies one of the equivalent conditions of Proposition 1.7.

We say that  $f$  is *regular along*  $L_{z_0}$  if it is regular at every  $(t, z_0)$ ,  $t \in T$ .

The following result is an immediate consequence of the condition (iii) of Proposition 1.7.

**Corollary 1.10.** *If  $f : (Y, y_0) \rightarrow (\mathbb{R}^k, 0)$ ,  $y_0 = \Phi(t_0, z_0)$ , is regular for  $\Phi$  then there is  $C > 0$  such that in a neighbourhood of  $(t_0, z_0)$*

$$(1.9) \quad \|f(\Phi(t, z)) - f(\Phi(t', z))\| \leq C\|t - t'\| \|f(\Phi(t_0, z))\|.$$

Regular functions have constant multiplicities along the fibers.

**Proposition 1.11.** *If  $f : (Y, y_0) \rightarrow (\mathbb{K}, 0)$ ,  $y_0 = \Phi(t_0, z_0)$ , is regular for  $\Phi$  then for  $t$  close to  $t_0$  the multiplicity of  $f_t(x) = f(t, x)$  at  $(t, z_0)$  is independent of  $t$ .*

*Proof.* We use the argument of Fukui's proof of invariance of multiplicity by blow-analytic homeomorphisms, cf. [13]. Note that

$$\text{mult}_0 f(x) = \min_{x(s)} \text{ord}_0 f(x(s)),$$

where the minimum is taken over all real analytic arcs  $x(s) : I \rightarrow (\mathbb{K}^n, 0)$ . For  $f$  regular these orders are preserved by  $\Phi$ .  $\square$

**Proposition 1.12.** *Let  $\Phi(t, z) : T \times Z \rightarrow Y$  be an arc-wise analytic trivialization and let  $f, g : (Y, y_0) \rightarrow (\mathbb{K}, 0)$  be two analytic function germs. Then  $f$  and  $g$  are regular if and only if so is  $fg$ .*

*Proof.* It follows from (ii) of Proposition 1.7.  $\square$

In the complex case the regularity is a geometric notion as the following proposition shows.

**Proposition 1.13.** *Suppose  $\mathbb{K} = \mathbb{C}$ . Let  $\Phi(t, z) : T \times Z \rightarrow Y$  be an arc-wise analytic trivialization and let  $f : Y \rightarrow \mathbb{C}$  be a complex analytic function. Suppose that  $\Phi$  preserves  $V(f)$ . Then  $f$  is regular for  $\Phi$  at every point of  $V(f)$ .*

*Proof.* Suppose that this is not the case. Then there exists a real analytic arc  $z(s) : (I, 0) \rightarrow (Z, z_0)$ , such that in (1.7),  $D_{k_0} \neq 0$  and  $D_{k_0}(t_0) = 0$ . Clearly  $f \circ \Phi(t_0, z(s)) \neq 0$  for  $s \neq 0$ . We show that for  $s \neq 0$  there is  $t(s)$ ,  $t(s) \rightarrow t_0$  as  $s \rightarrow 0$ , such that  $f \circ \Phi(t(s), z(s)) = 0$ . This would contradict the assumption on  $\Phi$ . For this, by restricting to a  $\mathbb{K}$ -analytic arc through  $t_0$ , we may suppose that  $t$  is a single variable  $t \in (\mathbb{C}, 0)$ . Let us then write  $f \circ \Phi(t, z(s)) = s^{k_0} h(t, s)$ , where

$$h(t, s) = D_{k_0}(t) + \sum_{k > k_0} D_k(t) s^{k-k_0}.$$

Since 0 is an isolated root of  $h(t, 0) = 0$ , Rouché's Theorem implies that  $h(t, s) = 0$  has roots also for  $s \neq 0$ .  $\square$

Suppose now that  $T, Z, Y$  are open subsets of  $\mathbb{K}^m, \mathbb{K}^n$  and  $\mathbb{K}^{n+m}$ , respectively. Let  $\Phi : T \times Z \rightarrow Y$  be an arc-wise analytic trivialization of the standard projection  $\pi : \mathbb{K}^{n+m} \rightarrow \mathbb{K}^m$ . Consider an ideal  $\mathcal{I} = (f_1, \dots, f_k)$  of  $\mathcal{O}_Y$  and denote by  $X = V(\mathcal{I})$  its zero set and by  $X_t$  the set  $X \cap \pi^{-1}(t)$ . Recall that for  $y \in X \subset \mathbb{C}^n$  the Zariski tangent space  $T_y X$  is the kernel of the differential  $D_y(f_1, \dots, f_k)$ .

**Proposition 1.14.** *Suppose every  $f_i, i = 1, \dots, k$ , is regular for  $\Phi$ . Then, for every  $y = \Phi(t, z)$ ,  $T_y X_t = \pi^{-1}(t) \cap T_y X$  and for  $z$  fixed,  $\dim_{\mathbb{K}} T_{\Phi(t, z)} X_t$  is independent of  $t$ . In particular,  $\text{Sing} X_t = \pi^{-1}(t) \cap \text{Sing} X$  and  $\Phi$  preserves  $\text{Sing} X$ .*

*Proof.* The equality  $T_y X_t = \pi^{-1}(t) \cap T_y X$  follows from the fact that the tangent space to the leaf through  $y$  satisfies  $T_y L_z \subset T_y X$  and is transverse to the fibers of  $\pi$ .

The differential of  $f$  at  $y$  vanishes if and only if

$$\min_{y(s)} \text{ord}_0 f(y(s)) > 1$$

where the minimum is taken over all real analytic arcs  $y(s) : I \rightarrow (Y, y)$ . Similarly, the differentials of  $f_1, \dots, f_l$  at  $y$  are independent if and only if for every  $i = 1, \dots, l$  there is a real analytic arc  $y(s) : I \rightarrow (Y, y)$  such that

$$\text{ord}_0 f_i(y(s)) = 1 \text{ and } \text{ord}_0 f_j(y(s)) > 1 \text{ for all } j = 1, \dots, \hat{i}, \dots, l.$$

All these conditions are preserved by  $\Phi$ . □

## 2. CONSTRUCTION OF ARC-WISE ANALYTIC TRIVIALIZATIONS

In this section we use Whitney Interpolation and Puiseux with parameter theorem to construct arc-wise analytic trivializations of equisingular (in the sense of Zariski) families of plane curve singularities. In Part 2 we will extend this construction to the Zariski equisingular families of hypersurface singularities of arbitrary number of variables.

Let

$$(2.1) \quad F(t, x, z) = z^N + \sum_{i=1}^N c_i(t, x) z^{N-i}$$

be a unitary polynomial in  $z \in \mathbb{K}$  with  $\mathbb{K}$ -analytic coefficients  $c_i(t, x)$  defined on  $U_{\varepsilon, r} = U_{\varepsilon} \times U_r$ , where  $U_{\varepsilon} = \{t \in \mathbb{K}^m; \|t\| < \varepsilon\}$ ,  $U_r = \{x \in \mathbb{K}; |x| < r\}$ . Here  $t$  is considered as a parameter. Suppose that the discriminant  $\Delta_F(t, x)$  of  $F$  is of the form

$$(2.2) \quad \Delta_F(t, x) = x^M u(t, x), \quad u \neq 0 \text{ on } U_{\varepsilon, r}.$$

If  $M = 0$  then by the Implicit Function Theorem the complex roots of  $F$ , denoted later by  $a_1(t, x), \dots, a_N(t, x)$ , are distinct  $\mathbb{K}$ -analytic functions of  $(t, x)$ . If  $M > 0$ , by Puiseux with parameter theorem they become analytic in  $(t, y)$  after a ramification  $x = y^d$ . Moreover these roots are distinct if  $x \neq 0$ . Hence, for  $x \in U_r \setminus \{0\}$  fixed, an ordering of the roots at  $(0, x)$ ,  $a_1(0, x), \dots, a_N(0, x)$ , gives a unique ordering of the roots at  $(t, x)$ ,  $a_1(t, x), \dots, a_N(t, x)$ , by continuity. By Corollary 2.3 this also holds for  $x = 0$ . Denote by  $a(t, x) = (a_1(t, x), \dots, a_N(t, x))$  the vector of these roots and consider the self map  $\Phi : U_{\varepsilon, r} \times \mathbb{C} \rightarrow U_{\varepsilon, r} \times \mathbb{C}$

$$(2.3) \quad \Phi(t, x, z) = (t, x, \psi(z, a(0, x), a(t, x))),$$

where  $\psi(z, a, b)$  is the Whitney interpolation map given by (9.14).

**Theorem 2.1.** *For  $\varepsilon > 0$  sufficiently small the map  $\Phi$  defined in (2.3) is an arc-wise analytic trivialization of the projection  $U_{\varepsilon,r} \times \mathbb{C} \rightarrow U_\varepsilon$ . It preserves the zero set of  $F$  and, moreover,  $F$  is regular along  $U_\varepsilon \times \{(0,0)\}$ .*

*If  $\mathbb{K} = \mathbb{R}$  then  $\Phi$  is conjugation invariant in  $z$ .*

Theorem 2.1 is shown in Subsection 2.2.

We denote by  $V(F) \subset U_{\varepsilon,r} \times \mathbb{C}$  the zero set of  $F$  and by  $V(F_0) \subset U_r \times \mathbb{C}$  the zero set of  $F|_{\{t=0\}}$ .

**2.1. Puiseux with parameter.** We recall the classical Puiseux with parameter theorem, see [63] Thm. 7 and [64] Thm. 4, also [51]. Puiseux with parameter theorem is a special case of the Abhyankar-Jung Theorem, see [1], [50].

**Theorem 2.2.** (Puiseux with parameter)

*Let  $F(t, x, z) \in \mathbb{C}\{t, x\}[z]$  be as in (2.1). Suppose that the discriminant of  $F$  is of the form  $\Delta_F(t, x) = x^M u(t, x)$  with  $u(0, 0) \neq 0$ . Then there is a positive integer  $d$  and  $\tilde{a}_i(t, y) \in \mathbb{C}\{t, y\}$  such that*

$$F(t, y^d, z) = \prod_{i=1}^N (z - \tilde{a}_i(t, y)).$$

*Let  $\theta$  be a  $d$ -th root of unity. Then for each  $i$  there is  $j$  such that  $\tilde{a}_i(t, \theta y) = \tilde{a}_j(t, y)$ .*

*If  $F(t, x, z) \in \mathbb{R}\{t, x\}[z]$  then the family  $\tilde{a}_i(t, y)$  is conjugation invariant.*

**Corollary 2.3.** *The roots of  $F$  at  $(t, 0)$ ,  $a_1(t, 0), \dots, a_N(t, 0)$ , are complex analytic functions in  $t$ . If  $a_i(0, 0) = a_j(0, 0)$  then  $a_i(t, 0) \equiv a_j(t, 0)$ . Thus the multiplicity of  $a_i(t, 0)$  as a root of  $F$  is independent of  $t$ .*

*Proof.* The family  $a_1(t, 0), \dots, a_N(t, 0)$  coincides with  $\tilde{a}_1(t, 0), \dots, \tilde{a}_N(t, 0)$ . If  $\tilde{a}_i(0, 0) = \tilde{a}_j(0, 0)$  then  $\tilde{a}_i(t, y) - \tilde{a}_j(t, y)$  divides  $y^{dM}$  and hence equals a power of  $y$  times a unit.  $\square$

**Corollary 2.4.** *The Puiseux pairs of  $a_i(t, x)$  and the contact exponents between different branches are independent of  $t$ .*

The following corollary is essential for the proof of Theorem 2.1. It allows to use the bi-lipschitz property given by Proposition 9.8.

**Corollary 2.5.**

$$(2.4) \quad \gamma(t, x) = \max_{a_i(t, x) \neq a_j(t, x)} \frac{|(a_i(t, x) - a_i(0, x)) - (a_j(t, x) - a_j(0, x))|}{|a_i(0, x) - a_j(0, x)|}$$

*is bounded by a power of  $\|t\|$  (with the exponent being a positive integer). Let  $a_i(t, x) \neq a_j(t, x)$ . Then*

$$(2.5) \quad \frac{a_i(t, x) - a_j(t, x)}{a_i(0, x) - a_j(0, x)} \rightarrow 1 \quad \text{as } t \rightarrow 0.$$

**2.2. Proof of Theorem 2.1.**  $\Phi$  is continuous by Proposition 9.9 and Remark 9.7. By Proposition 9.8 and Corollary 2.5, if  $\varepsilon$  is sufficiently small then for  $t$  and  $x$  fixed  $\psi_{a(0,x),a(t,x)} : \mathbb{C} \rightarrow \mathbb{C}$  is bi-Lipschitz. Therefore  $\Phi$  is bijective and the continuity of  $\Phi^{-1}$  follows from the invariance of domain.

**Lemma 2.6.** *For any  $r' < r$  there is  $C > 0$  such that the restriction  $\Phi : U_{\varepsilon,r'} \times \mathbb{C} \rightarrow U_{\varepsilon,r'} \times \mathbb{C}$  satisfies*

$$(2.6) \quad C^{-1}|F(0, x, z)| \leq |F(\Phi(t, x, z))| \leq C|F(0, x, z)|.$$

*Proof.* The Lipschitz constants of  $\psi_{a(0,x),a(t,x)} : \mathbb{C} \rightarrow \mathbb{C}$  and of its inverse can be chosen independent of  $(t, x) \in U_{\varepsilon,r'}$ . Let  $L$  be a common upper bound for these constants. Then, because  $\psi_{a(0,x),a(t,x)}(a_i(0, x)) = a_i(t, x)$ ,

$$(2.7) \quad L^{-1}|z - a_i(0, x)| \leq |\psi_{a(0,x),a(t,x)}(z) - a_i(t, x)| \leq L|z - a_i(0, x)|$$

Because  $F(\Phi(t, x, z)) = \prod_i (\psi_{a(0,x),a(t,x)}(z) - a_i(t, x))$  taking the product over  $i$  we obtain (2.6) with  $C = L^N$ .  $\square$

Let  $(x(s), z(s))$  be a real analytic arc and we assume that  $x(s)$  is not constant. Then the discriminant of  $F(t, x(s), z + z(s))$  equals  $s^{M_1}$  times an analytic unit. Suppose  $M_1 > 0$ . By replacing  $F$  by  $(t, s, z) \rightarrow F(t, x(s), z + z(s))$  we may assume that  $x(s) \equiv s$  and  $z(s) \equiv 0$ . After (9.14) we consider

$$(2.8) \quad \varphi(t, s) = \psi(0, a(s), b(t, s)) = \frac{\sum_k (\sum_{j=1}^N Q_{k,j}(a(s), b(t, s))) \overline{Q}_k(a(s))}{N! (\sum_k Q_k(a(s)) \overline{Q}_k(a(s)))},$$

where  $a(s) = a(0, s)$ ,  $b(t, s) = a(t, s)$ , and

$$Q_k(a) = \prod_i a_i^{N!} P_k(a_1^{-1}, \dots, a_N^{-1}),$$

$$Q_{k,j}(a, b) = \prod_i a_i^{N!} a_j^{-1} \frac{\partial P_k}{\partial \xi_j}(a_1^{-1}, \dots, a_N^{-1})(b_j - a_j).$$

Since  $P_k(a)$  is symmetric in  $a$ ,  $Q_k(a)$  is a polynomial in the coefficients  $c_i(0, s)$  of  $F$ . Hence  $Q_k(a(s))$  and  $\overline{Q}_k(a(s))$  are real analytic in  $s \in \mathbb{R}$ .

Next we study  $\sum_{j=1}^N Q_{k,j}(a(s), b(t, s))$ .

**Lemma 2.7.** *Let  $Q(a, b) \in \mathbb{C}[a, b]$  be a polynomial invariant under the action of the permutation group:  $Q(\sigma(a), \sigma(b)) = Q(a, b)$  for all  $\sigma \in S_N$ . Then  $Q(a(s), b(t, s)) \in \mathbb{C}\{t, s\}$ .*

*Proof.* By invariance under permutations  $Q(a(s), b(t, s))$  is well-defined for  $(t, s) \in B \times D$ , where  $B$  is a neighborhood of the origin in  $\mathbb{C}^m$  and  $D$  is a small disc centered at the origin in  $\mathbb{C}$ . Moreover it is bounded and complex analytic on  $B \times (D \setminus \{0\})$ . Therefore it is complex analytic on  $B \times D$ .  $\square$



Thus, by Lemma (2.7) the numerator of (2.8) is analytic in  $t \in \mathbb{K}^m$ ,  $s \in \mathbb{R}$ . As we have shown before its denominator is analytic in (one variable)  $s \in \mathbb{R}$ . Moreover  $\varphi(t, s)$  is bounded and therefore has to be analytic.

If  $M_1 = 0$ , then  $a(t, x(s))$  is analytic by the IFT.

If  $x(s) \equiv 0$  then consider  $f(t, z) = F(t, 0, z)$ . By Corollary 2.3 the number of complex roots of  $f$  is independent of  $t$  and hence the discriminant of the  $(f)_{red}$  does not vanish. Thus by the IFT,  $\Phi$  on  $x = 0$  is analytic.

The next lemma shows that the inverse of  $\Phi$  is arc-analytic and completes the proof of Theorem 2.1.

**Lemma 2.8.** *If  $(t(s), x(s), z(s))$  is a real analytic arc then there is a real analytic  $\tilde{z}(s)$  such that  $(t(s), x(s), z(s)) = \Phi(t(s), x(s), \tilde{z}(s))$ .*

*Proof.* Since  $\Phi^{-1}$  is subanalytic such  $\tilde{z}(s)$  exists continuous and subanalytic. Thus there is a positive integer  $q$  such that for  $s \geq 0$ ,  $\tilde{z}(s)$  is a convergent power series in  $s^{1/q}$ . We show that all exponents of  $\tilde{z}(s)$ ,  $s \geq 0$ , are integers. Suppose that this is not the case. Then

$$\tilde{z}(s) = \sum_{i=1}^n v_i s^i + v_{p/q} s^{p/q} + \sum_{k>p} v_{k/q} s^{k/q},$$

with  $p/q > n$  and  $p/q \notin \mathbb{N}$ . Denote  $\tilde{z}_{an}(s) = \sum_{i=1}^n v_i s^i$ . Then  $\psi(\tilde{z}_{an}(s), a(0, x(s), a(t(s), x(s))))$  is real analytic and by the bi-Lipschitz property, Proposition 9.8,

$$|\psi(\tilde{z}_{an}(s), a(0, x(s), a(t(s), x(s)))) - \psi(\tilde{z}(s), a(0, x(s), a(t(s), x(s))))| \sim |\tilde{z}_{an}(s) - \tilde{z}(s)| \sim s^{p/q}$$

that is impossible since  $\psi(\tilde{z}_{an}(s), a(0, x(s), a(t(s), x(s))))$  and  $\psi(\tilde{z}(s), a(0, x(s), a(t(s), x(s))))$  are real analytic in  $s$ .

This shows that  $(t(s), x(s), z(s))$ ,  $\Phi(t(s), x(s), \tilde{z}(s))$  are two real analytic arcs that coincide for  $s \geq 0$  and therefore for all  $s \in I$ .  $\square$

## Part 2. Zariski Equisingularity.

### 3. ZARISKI EQUISINGULARITY IMPLIES ARC-WISE ANALYTIC TRIVIALITY.

In this section we generalize Theorem 2.1 to the arbitrary number of variables hypersurface case.

**Definition 3.1.** By a *system of pseudopolynomials* in  $x = (x_1, \dots, x_n) \in \mathbb{K}^n$  (with parameters  $t \in (\mathbb{K}^m, 0)$ ) we mean a family  $F_i(t, x)$ ,  $i = 0, \dots, n$ , defined on  $U \times U_i$ , where  $U$  is a neighborhood of the origin in  $\mathbb{K}^m$ ,  $U_i$  is a neighbourhood of the origin in  $\mathbb{K}^i$ , of the form

$$(3.1) \quad F_i(t, x_1, \dots, x_i) = x_i^{d_i} + \sum_{j=1}^{d_i} A_{i,j}(t, x_1, \dots, x_{i-1}) x_i^{d_i-j}, \quad i = 1, \dots, n,$$

if  $d_i > 0$ , or if  $d_i = 0$ , then  $F_i$  is identically equal to 1, and by convention we define all  $F_j$ ,  $j < i$ , as identically equal to 1. For each  $i = 1, \dots, n$ , we assume that the discriminant

of  $F_{i,red}$  divides  $F_{i-1}$ , and that all the coefficients  $A_{i,j}$  are analytic and vanish identically on  $T = U \times \{0\}$ .

We consider the system  $\{F_i\}$  as a family function germs at the origin in  $\mathbb{K}^n$ , parameterized by the parameter  $t \in (\mathbb{K}^m, 0)$ . If  $F_0(0) \neq 0$  we call this system *Zariski equisingular*. As Varchenko showed in [54] (answering a question posed by Zariski in [65]), for a Zariski equisingular system, the family of analytic set germs  $X_t = \{F_n(t, x) = 0\} \subset (\mathbb{K}^n, 0)$  is topologically equisingular for  $t$  close to the origin. In this section we show that this equisingularity can be obtained by an arc-wise analytic trivialization.

*Remark 3.2.* Our definition is slightly more general than that of [65] or [54] where it is assumed that  $F_{i-1}$  is the Weierstrass polynomial associated to the discriminant of  $F_{i,red}$ . For the proof of Theorem 3.3, see below, we need in the inductive step that the discriminant of  $F_{i,red}$  is regular for the trivialization  $\Phi_{i-1}$ . By Proposition 1.7 this is the case if this discriminant divides  $F_{i-1}$  and  $F_{i-1}$  is regular for  $\Phi_{i-1}$ .

**Theorem 3.3.** *If  $F_i(t, x)$ ,  $i = 0, \dots, n$ , is a Zariski equisingular system of pseudopolynomials. then there exist  $\varepsilon > 0$  and a homeomorphism*

$$(3.2) \quad \Phi : B_\varepsilon \times \Omega_0 \rightarrow \Omega,$$

where  $B_\varepsilon = \{t \in \mathbb{K}^m; \|t\| < \varepsilon\}$ ,  $\Omega_0$  and  $\Omega$  are neighborhoods of the origin in  $\mathbb{K}^n$  and  $\mathbb{K}^{m+n}$  resp., such that

$$(Z1) \quad \Phi(t, 0) = (t, 0), \quad \Phi(0, x_1, \dots, x_n) = (0, x_1, \dots, x_n);$$

$$(Z2) \quad \Phi \text{ has a triangular form}$$

$$\Phi(t, x_1, \dots, x_n) = (t, \Psi_1(t, x_1), \dots, \Psi_{n-1}(t, x_1, \dots, x_{n-1}), \Psi_n(t, x_1, \dots, x_n));$$

$$(Z3) \quad \text{For } (t, x_1, \dots, x_{i-1}) \text{ fixed, } \Psi_i(t, x_1, \dots, x_{i-1}, \cdot) : \mathbb{K} \rightarrow \mathbb{K} \text{ is bi-Lipschitz and the Lipschitz constants of } \Psi_i \text{ and } \Psi_i^{-1} \text{ can be chosen independent of } (t, x_1, \dots, x_{i-1});$$

$$(Z4) \quad \Phi \text{ is an arc-wise analytic trivialization of the standard projection } \Omega \rightarrow B_\varepsilon;$$

$$(Z5) \quad F_n \text{ is regular along } B_\varepsilon \times \{0\}.$$

Recall after Proposition 1.12 that (Z5) implies that for any analytic  $G$  dividing a power of  $F_n$  there is  $C > 0$  such that

$$(3.3) \quad C^{-1}|G(0, x)| \leq |G(\Phi(t, x))| \leq C|G(0, x)|.$$

In particular  $\Phi$  preserves the levels of  $G$ .

*Remark 3.4.* Strategy of proof.

The functions  $\Psi_i$  will be constructed inductively so that every

$$(3.4) \quad \Phi_i(t, x_1, \dots, x_i) = (t, \Psi_1(t, x_1), \dots, \Psi_i(t, x_1, \dots, x_i))$$

satisfies the above properties (Z1)-(Z4) and (Z5) for  $F_i$ . The trivialization  $\Phi_{n-1}$  lifts (by continuity) to all (complex) roots of  $F_n$ , after which is extended to  $B \times \mathbb{K}^{n-1} \times \mathbb{C}$  by Whitney Interpolation Formula. The proof of the fact that the (obtained in this way) trivialization  $\Phi(t, x)$  is arc-wise analytic is based on the reduction to the Puiseux with parameter case.

Let  $x(s) = (x'(s), x_n(s))$  be a real analytic arc. By the inductive assumption  $\Phi_{n-1}(t, x'(s))$  is analytic in  $t, s$ . We show that  $f(t, s, z) = (F_n(\Phi_{n-1}(t, x'(s)), z))_{red}$  satisfies the assumptions of Puiseux with parameter theorem, and then we conclude by Theorem 2.1. For this, we first consider the case when the discriminant of  $F_{n,red}(\Phi_{n-1}(t, x'(s)), z)$  is not identically equal to zero, to show, for instance, that the number and the multiplicities of the roots of  $F_n$  are constant along each leaf of  $\Phi$ .

The property (5) of Definition 1.2 for  $\Phi$  will be shown later in section 5 where the appropriate stratification is introduced. In the argument below we do not use this property in the inductive step.

*Proof.* The proof is by induction on  $n$ . Thus suppose that  $\Psi_1, \dots, \Psi_{n-1}$  are already constructed and that for  $i < n$  the homeomorphisms (3.4) satisfy the properties (Z1)-(Z5). To simplify the notation we write  $(x_1, \dots, x_n) = (x', x_n)$ . Let  $\Phi_{n-1} : B_{\varepsilon'} \times \Omega'_0 \rightarrow \Omega'$ . By the inductive assumption  $F_{n-1}$ , and hence by Proposition 1.12 the discriminant of  $F_{n,red}$ , is regular for  $\Phi_{n-1}$ .

For a real analytic arc  $x'(s) : I \rightarrow \Omega'_0$  we consider  $F_n$  over the flow  $\Phi_{n-1}(t, x'(s))$  and apply to it Theorem 2.1.

**Lemma 3.5.** *Let  $x'(s) : I \rightarrow \Omega'_0$ ,  $x'(0) = x'_0$ , be a real analytic arc that is not entirely included in the zero set of  $F_{n-1}$ . Then the discriminant  $\Delta(t, s)$  of  $(t, s, z) \rightarrow F_{n,red}(\Phi_{n-1}(t, x'(s)), z)$  is of the form  $s^M$  times an analytic unit.*

*Proof.* By (Z4) of the inductive assumption,  $\Phi_{n-1}(t, x'(s))$  is analytic in  $t, s$  and  $\Delta(t, s)$  is not identically equal to zero and divides  $F_{n-1}(\Phi_{n-1}(t, x'(s)))$ . The zero set of  $F_{n-1}(\Phi_{n-1}(t, x'(s)))$  is  $s = 0$ , or the empty set, and hence, by (3.3) for  $F_{n-1}$ ,  $F_{n-1}$ , and hence  $\Delta(t, s)$  as well, is of the form  $s^M$  times an analytic unit.  $\square$

**Lemma 3.6.** *Fix  $x'_0 \in \Omega'_0$ . Then, the number and the multiplicities of the complex roots of  $F_n(\Phi_{n-1}(t, x'_0), z)$ , considered as a polynomial in  $z \in \mathbb{C}$ , are independent of  $t$  and these roots depend  $\mathbb{K}$ -analytically on  $t$ .*

*Proof.* Denote

$$(3.5) \quad f(t, z) = F_n(\Phi_{n-1}(t, x'_0), z) = z^{d_n} + \sum_i B_i(t) z^{d_n-i}$$

By the inductive assumption (Z4) on  $\Phi_{n-1}$ , the coefficients of  $B_i(t)$  are  $\mathbb{K}$ -analytic in  $t$ . If  $F_{n-1}(0, x') \neq 0$ , then it follows from the assumption (Z5) for  $F_{n-1}$  that the discriminant  $\Delta(t)$  of  $f_{red}(t, z)$  does not vanish on  $t \in B_{\varepsilon'}$ . Therefore by the IFT, Lemma 9.14, the roots of  $f_{red}$ , are  $\mathbb{K}$ -analytic in  $t$ , distinct for all  $t \in B'_{\varepsilon}$ , and the multiplicities of these roots as the roots of  $f$  must be independent of  $t$ .

If  $F_{n-1}(0, x'_0) = 0$ , we choose a real analytic arc germ  $x'(s) : I \rightarrow \mathbb{K}^{n-1}$ ,  $x'(0) = x'_0$  that is not entirely included in the zero set of  $F_{n-1}$ . By Lemma 3.5 we may apply to  $(t, s) \rightarrow F_{n,red}(\Phi_{n-1}(t, x'(s)), z)$  the Puiseux with parameter theorem and the claim now follows from Corollary 2.3.  $\square$

*Remark 3.7.* Lemma 3.6 holds under slightly more general assumptions. We only need that  $F_n \in \mathbb{K}\{t, x'\}[x_n]$  is a monic polynomial and that  $F_{n-1}(t, x')$  is regular for the trivialization  $\Phi'$  and that the discriminant of  $F_{n,red}$  divides  $F_{n-1}$ .

Denote by

$$a(t, x') = (a_1(t, x'), \dots, a_{d_n}(t, x')),$$

the vector of complex roots of  $F_n$  depending  $\mathbb{K}$ -analytically on  $t$ . We set

$$(3.6) \quad \begin{aligned} \Psi_n(t, x) &:= \psi(x_n, a(0, x'), a(\Phi_{n-1}(t, x'))) \\ &= x_n + \frac{\sum_{j=1}^N \mu_j(x_n, a(0, x'))(a_j(\Phi_{n-1}(t, x')) - a_j(0, x'))}{\sum_{j=1}^N \mu_j(x_n, a(0, x'))}, \end{aligned}$$

where  $\psi$  is given by (9.14), and then define  $\Phi$  by (Z2). Thus  $\Phi$  satisfies (Z1). We show that  $\Phi$  is a homeomorphism that satisfies (Z3)-(Z5). This we check on every real analytic arc applying Puiseux with parameter theorem.

**Lemma 3.8.** *Let  $K \subseteq \Omega'_0$ . Then*

$$(3.7) \quad \sup_{x' \in K} \max_{a_i(0, x') \neq a_j(0, x')} \frac{|a_i(\Phi_{n-1}(t, x')) - a_j(\Phi_{n-1}(t, x'))|}{|a_i(0, x') - a_j(0, x')|} \rightarrow 1 \quad \text{as } t \rightarrow 0$$

*Proof.* Denote

$$\gamma(t, x') = \max_{a_i(0, x') \neq a_j(0, x')} \frac{|(a_i(\Phi_{n-1}(t, x')) - a_j(\Phi_{n-1}(t, x')) - (a_i(0, x') - a_j(0, x'))|}{|a_i(0, x') - a_j(0, x')|}.$$

We show that  $\gamma$  is bounded on  $B_{\varepsilon'} \times K$  and converges to 0 as  $t$  goes to 0. Let  $x'(s)$  be a real analytic arc such that  $(0, x'(s))$  is not entirely included in the zero set of  $F_{n-1}$ . By Corollary 2.5,  $\gamma$  is bounded on  $(t, x'(s))$  and converges to 0 as  $t$  goes to 0. Thus, by the curve selection lemma, the claim holds on  $\{(t, x'); F_{n-1}(t, x') \neq 0\}$ . We extend it on the zero set of  $F_n$  by the lower semi-continuity of  $\gamma$ , Remark 9.10.  $\square$

Thus, taking  $\varepsilon$  smaller if necessary, we see by Proposition 9.8 that  $\Psi_n$  of (3.6) is well-defined, continuous by Proposition 9.9, and satisfies (Z3).

Now we show (Z4) (except the property (5) of Definition 1.2 that will be shown in section 5). Let  $x(s) : I \rightarrow \Omega_0$  be a real analytic arc. We show that  $\Phi(t, x(s))$  is  $\mathbb{K}$ -analytic in  $t$  and real analytic in  $s$ . If  $(0, x'(s))$  is not entirely included in the zero set of  $F_{n-1}$  then it follows from Lemma 3.5 and Theorem 2.1. Thus, suppose  $F_{n-1}(0, x'(s)) \equiv 0$ . Consider

$$(3.8) \quad f(t, s, z) = (F_n(\Phi_{n-1}(t, x'(s)), z))_{red}.$$

By (3.7) and Lemma 3.6, the discriminant of  $f$  is of the form  $s^M$  times a unit, thus again the claim follows from Theorem 2.1. To show that the inverse of  $\Phi$  is arc-analytic we use the inductive assumption, i.e. the inverse of  $\Phi_{n-1}$ , is arc-analytic, and then, for a real analytic arc  $x'(s)$ , over  $(t, s) \rightarrow \Phi_{n-1}(t, x'(s))$ , we use Lemma 2.8.

The proof of (Z5) is similar to the proof of (Z4). First, by Proposition 1.7, it suffices to show it over the flow of any real analytic curve  $x'(s)$ , that is for  $(t, s) \rightarrow \Phi_{n-1}(t, x'(s))$ . If

$(0, x'(s))$  is not entirely included in the zero set of  $F_{n-1}$ , then it follows directly from Lemma 3.5 and Theorem 2.1. If  $F_{n-1}(0, x'(s)) \equiv 0$  then we consider (3.8) and conclude again by Theorem 2.1.  $\square$

**3.1. Geometric properties.** In this subsection we summarize some geometric properties of the arc-wise analytic trivializations constructed in the proof of Theorem 3.3.

The following proposition is a consequence of Proposition 1.11. It also follows from [66].

**Proposition 3.9** (Zariski equisingularity implies equimultiplicity). *Let  $F_i$ ,  $i = 0, \dots, n$ , be a Zariski equisingular system of analytic functions at the origin in  $\mathbb{K}^m \times \mathbb{K}^n$ . Then for any  $\mathbb{K}$ -analytic function  $G$  dividing  $F_n$ , the multiplicity at  $(t, 0)$  of  $G_t(x) = G(t, x)$  is independent of  $t$ .*  $\square$

By construction  $\Phi(t, x) = (t, \Psi(t, x))$  is real analytic in the complement of  $B_\varepsilon \times Z$ , where  $Z$  is a nowhere dense  $\mathbb{K}$ -analytic subset of  $\Omega_0$ . Let us for  $t$  fixed denote  $x \rightarrow \Psi(t, x)$  by  $\Psi_t$ . It follows from (Z2) and (Z3) that the jacobian determinant of  $\Psi_t$ , that is well-defined in the complement of  $B_\varepsilon \times Z$ , is bounded from zero and infinity in a neighborhood of the origin, that is there exists  $C, c > 0$  such that

$$(3.9) \quad c \leq |\text{jac det}(\Psi_t)(t, x)| \leq C.$$

Then, by the proof of Proposition 1.7 that can be applied to  $f = \text{jac det}(\Psi_t)$ , the other two conditions of Proposition 1.7 hold.

Consider an analytic set  $X = \{f_1(t, x) = \dots = f_k(t, z) = 0\} \subset \Omega$  defined by  $\mathbb{K}$ -analytic functions  $f_1(t, x), \dots, f_k(t, z)$  regular for  $\Phi$ . Denote  $X_t = X \cap \pi^{-1}(t)$ . Then, as follows from Proposition 1.14,  $\text{Sing}X_t = \pi^{-1}(t) \cap \text{Sing}X$  and  $\Phi$  preserves  $\text{Sing}X$  and  $\text{Reg}X$ .

**3.2. Generalizations.** The following generalization can be used to show the topological equisingularity of analytic function germs, see [3] and subsection 8.3 below.

**Proposition 3.10.** *Theorem 3.3 holds if in the definition of a system of polynomials the assumption*

(i) *The discriminant of  $F_{i,\text{red}}$  divides  $F_{i-1}$ .*

*is replaced by*

(iii) *There are  $q_i \in \mathbb{N}$  such that  $F_i = x_1^{q_i} \tilde{F}_i$ , where  $\tilde{F}_i(x_1, \dots, x_i)$  is a monic Weierstrass polynomial in  $x_i$ , and for  $i = 1, \dots, n$ , the discriminant of  $\tilde{F}_{i,\text{red}}$  divides  $F_{i-1}$ .*

*Moreover, in the conclusion we may require that  $\Psi_1(t, x_1) \equiv x_1$ .*

*Proof.* We can always require  $\Psi_1(t, x_1) \equiv x_1$  as it follows from the construction. Then  $x_1$  is constant on the fibers of  $\Phi$  and hence regular along  $B_\varepsilon \times \{0\}$ . Therefore, under the above assumptions, if we choose  $\Psi_1(t, x_1) \equiv x_1$ , then in the inductive step if  $\Phi_i$  satisfies the properties (Z1)-(Z4) and (Z5) for  $\tilde{F}_i$ , then it satisfies them for  $F_i = x_1^{q_i} \tilde{F}_i$ .  $\square$

## 4. ZARISKI EQUISINGULARITY WITH TRANSVERSE PROJECTIONS.

**Definition 4.1.** We say that a system of pseudopolynomials  $F_i(t, x)$ ,  $i = 1, \dots, n$ , is *transverse* at the origin in  $\mathbb{K}^m \times \mathbb{K}^n$ , if for each  $i = 2, \dots, n$ , the multiplicity  $\text{mult}_0 F_i(0, x)$  of  $F_i(0, x)$  at  $0 \in \mathbb{K}^i$  is equal to  $d_i$ .

We always have the upper semi-continuity condition. If we denote  $F_t(x) = F(t, x)$ , then  $\text{mult}_0 F_t \leq \text{mult}_0 F_0$  for  $t$  close to 0. Therefore the transversality is a closed condition (in euclidean or analytic Zariski topology) in parameter  $t$ . If the system  $\{F_i\}$  is Zariski equisingular then, by Proposition 3.9, the transversality is also an open condition.

Denote  $X = F_n^{-1}(0)$ ,  $X_t = X \cap \{t\} \times \mathbb{K}^n$ . Geometrically the assumption  $\text{mult}_0 F_n(0, x) = d_n$  means that the kernel of the standard projection  $\mathbb{K}^n \rightarrow \mathbb{K}^{n-1}$  is transverse to the tangent cone of  $X_0$  at the origin. Moreover, in the Zariski equisingular case, by the equimultiplicity, the kernel of the standard projection  $\pi : \mathbb{K}^m \times \mathbb{K}^n \rightarrow \mathbb{K}^m \times \mathbb{K}^{n-1}$  is then transverse to the tangent cone of  $X$  at the origin.

**Theorem 4.2.** Let  $F_i(t, x)$ ,  $i = 0, \dots, n$ , be a Zariski equisingular system of pseudopolynomials transverse at the origin in  $\mathbb{K}^m \times \mathbb{K}^n$ . Let  $\Phi(t, x) = (t, \Psi(t, x)) : B_\varepsilon \times \Omega_0 \rightarrow \Omega$  be a homeomorphism as constructed in the proof of Theorem 3.3. Then

(Z6)  $\Phi$  is an arc-wise analytic trivialization regular along  $B_\varepsilon \times \{0\}$ .

Thus (Z6) means that there is a constant  $C > 0$  such that for all  $(t, x) \in B_\varepsilon \times \Omega_0$

$$(4.1) \quad C^{-1}\|x\| \leq \|\Psi(t, x)\| \leq C\|x\|$$

$$(4.2) \quad \left\| \frac{\partial \Psi}{\partial t}(t, x) \right\| \leq C\|\Psi(t, x)\|$$

Theorem 4.2 follows from Proposition 1.6 and the following result.

**Lemma 4.3.** Let  $F_i(t, x)$ ,  $i = 0, \dots, n$ , be a Zariski equisingular system of pseudopolynomials transverse at the origin in  $\mathbb{K}^m \times \mathbb{K}^n$ . Let  $\Phi : B_\varepsilon \times \Omega_0 \rightarrow \Omega$  be a homeomorphism as constructed in the proof of Theorem 3.3. Then for every real analytic arc  $x(s) : I \rightarrow \Omega_0$ , such that  $x(0) = 0$ , the leading coefficient  $D_{k_0}$  of

$$(4.3) \quad (\Psi_1(t, x_1(s)), \dots, \Psi_n(t, x_1(s), \dots, x_n(s))) = \sum_{k \geq k_0} D_k(t) s^k$$

does not vanish on  $B_\varepsilon$ .

*Proof.* Induction on  $n$ . Let  $x(s) = (x'(s), x_n(s))$ . Let us write for short

$$\Phi(t, x) = (t, \Psi(t, x)) = (t, \Psi'(t, x), \Psi_n(t, x)).$$

By the inductive assumption

$$(4.4) \quad \Psi'(t, x'(s)) = \sum_{k \geq k'_0} B_k(t) s^k$$

with  $B_{k'_0}(t)$  not vanishing on  $B_\varepsilon$ .

First consider the case  $x_n(s) = a_i(0, x'(s))$ . Then

$$\Phi_n(t, x'(s), a_i(0, x'(s))) = (\Phi_{n-1}(t, x'(s)), a_i(\Phi_{n-1}(t, x'(s)))).$$

We claim that there are constants  $C_1, C_2$  such that for every  $t \in B_\varepsilon$

$$(4.5) \quad \begin{aligned} \max(\|x'(s)\|, \max_i |a_i(0, x'(s))|) &\leq C_1 \max(\|\Psi'(t, x'(s))\|, \max_i |a_i(\Phi_{n-1}(t, x'(s)))|) \\ &\leq C_2 \max(\|x'(s)\|, \max_i |a_i(0, x'(s))|). \end{aligned}$$

In other words we claim that the size of  $\|\Psi'(t, x'(s))\| + \sum_i |a_i(\Phi_{n-1}(t, x'(s)))|$  (up to a constant) is independent of  $t$ . The analogous property for  $\|\Psi'(t, x'(s))\|$  follows from the inductive assumption. For  $\max_{i \neq j} |a_i(\Phi_{n-1}(t, x'(s))) - a_j(\Phi_{n-1}(t, x'(s)))|$  it follows from (2.5). Now, for any numbers  $a_i$ ,  $i = 1, \dots, d$ ,

$$\max_i |a_i| \leq \max(|\sum_i a_i|, \max_{i \neq j} |a_i - a_j|) \leq d \max_i |a_i|.$$

Moreover,

$$|\sum_i a_i(\Phi_{n-1}(t, x'(s)))| = |A_{d-1,1}(\Phi_{n-1}(t, x'(s)))| \leq C \|x'(s)\|,$$

since  $A_{d-1,1}$  is an analytic function vanishing identically of  $B_\varepsilon \times \{0\}$ . Thus the size of  $\|\Psi'(t, x'(s))\| + \sum_i |a_i(\Phi_{n-1}(t, x'(s)))|$  is equivalent (up to a constant) to the size of  $\|\Psi'(t, x'(s))\| + \sum_{i \neq j} |a_i(\Phi_{n-1}(t, x'(s))) - a_j(\Phi_{n-1}(t, x'(s)))|$ , and this shows (4.5).

Therefore, for every  $j$

$$(4.6) \quad |a_j(\Phi_{n-1}(t, x'(s)))| \leq C_2 \max(\|x'(s)\|, \max_i |a_i(0, x'(s))|) \leq C_3 \|x'(s)\|,$$

where the last inequality follows from the transversality assumption  $\text{mult}_0 F_n(0, x) = d_n$ . This shows the claim with  $k_0 = k'_0$ .

Suppose now that  $x_n(s)$  is arbitrary. By Lipschitz property, Proposition 9.8,

$$|\Psi_n(t, x(s)) - a_i(\Phi_{n-1}(t, x'(s)))| \sim |x_n(s) - a_i(0, x'(s))|$$

and therefore

$$\Psi_n(t, x(s)) - a_i(\Phi_{n-1}(t, x'(s))) = \sum_{k \geq l} b_k(t) s^k,$$

with  $b_l(t)$  that does not vanish on  $B_\varepsilon$ . Thus there are two possible cases. If  $l \geq k'_0$  then  $k_0 = k'_0$ . If  $l < k'_0$  then  $k_0 = l$ .

This ends the proof of Lemma 4.3 and Theorem 4.2.  $\square$

## 5. STRATIFICATION ASSOCIATED TO A SYSTEM OF PSEUDOPOLYNOMIALS.

In this section we consider a classical construction of a stratification associated to a system of polynomials and show that the trivialization  $\Phi$  constructed in the proof of Theorem 3.3 preserves this stratification and hence satisfies condition (5) of Definition 1.2.

Consider a *system of pseudopolynomials* in  $x = (x_1, \dots, x_n) \in \mathbb{K}^n$

$$(5.1) \quad F_i(x_1, \dots, x_i) = x_i^{d_i} + \sum_{j=1}^{d_i} A_{i-1,j}(x_1, \dots, x_{i-1}) x_i^{d_i-j}, \quad i = 1, \dots, n,$$

with  $\mathbb{K}$ -analytic coefficients  $A_{i-1,j}$ , satisfying

- (1) there are positive reals  $\varepsilon_j > 0$ ,  $j = 1, \dots, n$ , such that  $F_i$  are defined on

$$(5.2) \quad \Omega_i := \{(x_1, \dots, x_i) \in \mathbb{K}^i; |x_j| < \varepsilon_j, j = 1, \dots, i\} = \prod_{j=1}^i U_j$$

where  $U_j = \{|x_j| < \varepsilon_j\}$ .

- (2)  $F_i$  does not vanish on  $\Omega_{i-1} \times \partial U_i$ , where  $\partial U_i = \{|x_i| = \varepsilon_i\}$ .

- (3) for every  $i$ ,  $F_{i-1}$  divides the discriminant of  $F_{i,red}$ .

It may happen that  $d_i = 0$ . Then  $F_i \equiv 1$  and we set by convention  $F_j \equiv 1$  for  $j < i$ . For  $i < k$  we denote by  $\pi_{k,i} : \Omega_k \rightarrow \Omega_i$  the standard projection.

Fix  $i = 1, \dots, n$  and define  $\Omega'_{i,i} = \Omega_i \setminus V(F_i)$ . If  $i < n$  then for  $k > i$  we define recursively  $\Omega'_{i,k+1} = \pi_{k+1,k}^{-1}(\Omega'_{i,k}) \cap V(F_{k+1})$ . We shall need the following lemma.

**Lemma 5.1.** *Either  $\Omega'_{i,k}$  is empty or the map*

$$\Omega'_{i,k} \rightarrow \Omega'_{i,i},$$

*induced by  $\pi_{k,i}$  is a finite  $\mathbb{K}$ -analytic covering.*

*Proof.* Induction on  $k$ . For  $k = i + 1$  it follows from the implicit function theorem, see Lemma 9.14.

Suppose that lemma holds for  $k \leq j$ . We show it for  $k = j + 1$ . We localize at  $p_i \in \Omega'_i$  and  $p \in \Omega'_{i,j}$  such that  $\pi_{j,i}(p) = p_i$ . Let  $y_1, \dots, y_i$  be a local system of coordinates at  $p_i$ . By inductive assumption,  $\Omega'_{i,j}$  near  $p$  is the graph of a  $\mathbb{K}$ -analytic map  $(\varphi_{i+1}(y_1, \dots, y_i), \dots, \varphi_j(y_1, \dots, y_i))$ . After a change of coordinates  $y_l = x_l - \varphi_l(y_1, \dots, y_i)$ ,  $l = i+1, \dots, j$  the new system of functions

$$f_k(y_1, \dots, y_k) = F_j(y_1, \dots, y_i, y_{i+1} + \varphi_{i+1}(y_1, \dots, y_i), \dots, y_k + \varphi_k(y_1, \dots, y_i)),$$

is Zariski equisingular with  $y_1, \dots, y_i$  as parameters. Lemma 5.1 now follows from Lemma 3.6 and Remark 3.10.  $\square$

For every  $i$  we define a filtration

$$(5.3) \quad \Omega_i = X_i^i \supset X_{i-1}^i \supset \dots \supset X_0^i,$$

where

- (1)  $X_0^1 = V(F_1)$ . It may be empty.



$$(2) X_j^i = (\pi_{i,i-1}^{-1}(X_j^{i-1}) \cap V(F_i)) \cup \pi_{i,i-1}^{-1}(X_{j-1}^{i-1}).$$

Filtration 5.3 defines a stratification  $\mathcal{S}_i$  of  $\Omega_i$  by taking the connected components of all  $X_j^i \setminus X_{j-1}^i$  as strata.

**Corollary 5.2.** *For all  $j \leq i$  every connected component  $S$  of  $X_j^i \setminus X_{j-1}^i$  is a locally closed  $j$ -dimensional  $\mathbb{K}$ -analytic submanifold of  $\Omega_i$  of one of two types:*

- (I)  $S \subset V(F_i)$  and there is a connected component  $S'$  of  $X_j^{i-1} \setminus X_{j-1}^{i-1}$  such that  $\pi_{i,i-1}$  induces a finite  $\mathbb{K}$ -analytic covering  $S \rightarrow S'$ .
- (II) There is a connected component  $S''$  of  $X_{j-1}^{i-1} \setminus X_{j-2}^{i-1}$  such that  $S$  is a connected component of  $\pi_{i,i-1}^{-1}(S'') \setminus V(F_i)$ .

Moreover, for every  $i = 1, \dots, n-1$  and  $j \leq i$  the number and the multiplicities of complex roots of  $F_{i+1}$  considered as a polynomial in  $x_{i+1}$  are constant on connected components of  $X_j^i \setminus X_{j-1}^i$ .

*Proof.* The first claim follows from Lemma 5.1 and the second one from Lemma 3.6 and Remark 3.7.  $\square$

**5.1.  $\Phi$  of the proof of Theorem 3.3 satisfies condition (5) of Definition 1.2.** Let  $\mathcal{S}, \mathcal{S}_0$  be the associated stratifications of the families  $\{F_i(t, x)\}$  and  $\{F_i(0, x)\}$  respectively. We show that  $\Phi$  is real analytic diffeomorphism between the strata of  $B \times \mathcal{S}_0$  and  $\mathcal{S}$ . By induction on  $n$  we may suppose that the corresponding property holds for  $\Phi_{n-1}$ . Let  $S$  be a stratum of  $\mathcal{S}$  of type (I), that is a covering space over a stratum  $S'$ . Denote  $S_0 = S \cap \{t = 0\}$ ,  $S'_0 = S' \cap \{t = 0\}$ . Then if  $\Phi' : T \times S'_0 \rightarrow S'$  is analytic diffeomorphism so is its lift  $\Phi : T \times S_0 \rightarrow S$ .

Now suppose that  $S$  is of type (II). By assumption,  $a_i(t, x')$  of (3.6) are real analytic on  $B \times S'$  and hence, by Whitney interpolation formula (9.14), so is  $\Psi_n$  on  $B \times S''$ . This shows the claim.  $\square$

We complete this section by showing further properties of stratifications  $\mathcal{S}_i$ . For the definition of an arc-wise analytically trivial stratification see Subsection 7.1 and for condition  $(a_f)$  Subsection 8.2.

**Theorem 5.3.** *The stratification  $\mathcal{S}_i$  of  $\Omega_i$  is an arc-wise analytically trivial stratification of  $\Omega_i$  and satisfies condition  $(a_f)$  for  $F_i$ .*

*Proof.* Let  $\mathcal{S}_i$  be the stratification of  $\Omega_i$  defined by (5.3). Thus for each  $S \in \mathcal{S}_n$   $\dim S \geq 1$ , there is a unique  $i$  and a unique stratum  $S_i \subset \Omega_i \setminus V(F_i)$  such that  $S$  is a finite  $\mathbb{K}$ -analytic covering over  $S_i$ . If  $S_i$  is a connected component of  $\Omega_i \setminus V(F_i)$  then, by the proof of Lemma 5.1,  $\{F_j\}_{j \geq i}$  defines a Zariski equisingular system at every point of  $S$  and hence  $\mathcal{S}$  is arc-wise analytically trivial along such  $S$ .

Thus suppose the opposite, that  $\dim S_i < i$ . Fix  $p \in S$  and let  $p_i = \pi_{n,i}(p)$ . Let  $B_i$  be a neighborhood of  $p_i$  in  $\Omega_i \setminus V(F_i)$  and let  $B$  be a neighborhood of  $p$  in  $\Omega'_{n,i}$  such that  $\pi_{n,i}$

gives a diffeomorphism of  $B$  onto  $B_i$ .

$$\begin{array}{ccccc}
 S \times B_0 \times \Omega_0 & \longrightarrow & B \times \Omega_0 & \xrightarrow{\Phi} & \Omega \\
 \downarrow & & \downarrow & \swarrow & \\
 S \times B_0 & \xrightarrow{\varphi} & B & & \\
 \downarrow & \swarrow & & & \\
 S & & & & 
 \end{array}
 \tag{5.4}$$

Then we may again apply the argument of proof of Lemma 5.1 to construct an arc-wise analytic trivialization  $\Phi$  of a neighborhood  $\Omega$  of  $p$  along  $B$ . Similarly, by inductive assumption, we have an arc-wise analytic trivialization  $\varphi$  of  $B'$  along  $S$ . Thus we have the diagram (5.4), where we identify  $B$  and  $B'$ ,  $\tilde{S}$  denote a small neighborhood of  $p$  in  $S$  that we identify with its image in  $S_i$ , and  $\Omega_0 = \pi_{n,i}^{-1}(p_i)$ . The required arc-wise analytic trivialization  $\Gamma : S \times B_0 \times \Omega_0 \rightarrow \Omega$  is then given by  $\Gamma(s, b, x) = \Phi(\varphi(s, b), x)$ .  $\square$

## Part 3. Applications.

### 6. GENERIC ARC-WISE ANALYTIC EQUISINGULARITY

We use Zariski equisingularity to show that an analytic family of analytic set germs  $\mathcal{X} = \{X_t\}$ ,  $t \in T$ , is "generically" equisingular. That is, locally on the parameter space  $T$ , the family is equisingular in the complement of an analytic subset  $Z \subset T$ ,  $\dim Z < \dim T$ . In this section the parameter space  $T$  may be singular.

**Definition 6.1.** Let  $T$  be a  $\mathbb{K}$ -analytic space,  $U \subset \mathbb{K}^n$  an open neighborhood of the origin,  $\pi : T \times U \rightarrow T$  the standard projection, and let  $\mathcal{X} = \{X_k\}$  be a finite family of analytic subsets of  $T \times U$ . We say that  $\mathcal{X}$  is *arc-wise analytically equisingular along  $T \times \{0\}$  at  $t \in \text{Reg}(T)$* , if there are neighborhoods  $B$  of  $t$  in  $\text{Reg}(T)$  and  $\Omega$  of  $(t, 0)$  in  $T \times \mathbb{K}^n$ , and an arc-wise analytic trivialization  $\Phi : B \times \Omega_t \rightarrow \Omega$ , where  $\Omega_t = \Omega \cap \pi^{-1}(t)$ , such that  $\Phi(B \times \{0\}) = (B \times \{0\})$  and for every  $k$ ,  $\Phi(T \times X_{k,t}) = X_k$ , where  $X_{k,t} = X_k \cap \pi^{-1}(t)$ .

We say that  $\mathcal{X}$  is *regularly arc-wise analytically equisingular along  $T \times \{0\}$  at  $t \in T$*  if moreover  $\Phi$  is regular at  $(t, 0)$ .

**Theorem 6.2.** Let  $\mathcal{X} = \{X_k\}$  be a finite family of analytic subsets of a neighborhood of  $T \times U$  and let  $t_0 \in T$ . Then there exist an open neighborhood  $T'$  of  $t_0$  in  $T$  and a proper  $\mathbb{K}$ -analytic subset  $Z \subset T'$ , containing  $\text{Sing}(T')$ , such that for every  $t \in T' \setminus Z$ ,  $\mathcal{X}$  is regularly arc-wise analytically equisingular along  $T \times \{0\}$  at  $t$ .

Moreover, there is an analytic stratification of an open neighborhood of  $t_0$  in  $T$  such that for every stratum  $S$  and every  $t \in S$ ,  $\mathcal{X}$  is regularly arc-wise analytic equisingular along  $S \times \{0\}$  at  $t$ .

*Proof.* For each  $X_k$  fix a finite system of generators  $F_{k,i} \in \mathcal{O}_{T,t_0}$  of the ideal defining it. The first claim follows then from Lemma 6.3 applied to the product of all  $F_{k,i}$ . The second claim follows from the induction on  $\dim T$ .

**Lemma 6.3.** *Let  $T$  be a  $\mathbb{K}$ -analytic space,  $t_0 \in T$ . Let  $F$  be a  $\mathbb{K}$ -analytic function defined in a neighborhood of  $(t_0, 0) \in T \times \mathbb{K}^n$ . Then there exist a neighborhood  $T'$  of  $t_0$  in  $T$  and a proper  $\mathbb{K}$ -analytic subset  $Z \subset T'$ ,  $\dim Z < \dim T$ , such that, after a linear change of coordinates in  $\mathbb{K}^n$ , for every  $t \in T' \setminus Z$  there is a Zariski equisingular transverse system of functions  $F_i$ ,  $i = 0, \dots, n$ , at  $(t, 0)$ , with  $F_n$  the Weierstrass polynomial associated to  $F$  at  $(t, 0)$ .*

*Proof.* We may suppose that  $T$  is a subspace of  $\mathbb{K}^m$ ,  $t_0 = 0$ , and  $(T, 0)$  is irreducible.

We construct a new system of coordinates  $x_1, \dots, x_n$  on  $\mathbb{K}^n$ , analytic subspaces  $(Z_i, 0) \subset (T, 0)$  and analytic function germs  $G_i(t, x_1, \dots, x_i)$ ,  $i = n, n-1, \dots, 0$ , such that for every  $t \in T \setminus Z$ ,  $Z = \text{Sing}(T) \cup \bigcup Z_i$ , the following condition is satisfied. Let  $F_i$  be the Weierstrass polynomial in  $x_i$  associated to the germ of  $G_i$  at  $(t, 0)$ . Then the discriminant of  $F_{i,\text{red}}$  divides  $F_{i-1}$ .

The  $G_i$  are constructed by descending induction. First we set  $G_n = F$ . Then we construct  $G_{n-1}$  in three steps.

*Step 1.* Write

$$G_n(t, x) = \sum_{|\alpha| \geq m_0} A_\alpha(t) x^\alpha,$$

where  $m_0$  is the minimal integer  $|\alpha|$  for which  $A_\alpha \not\equiv 0$ . We may assume  $m_0 > 0$  otherwise we simply take  $Z = \emptyset$ . After a linear change of  $x$ -coordinates we may assume  $A(t) = A_{(0, \dots, 0, m_0)}(t) \not\equiv 0$ .

*Step 2.* If  $A(0) = 0$  then define  $A(t) * x := (A(t)^2 x_1, \dots, A(t)^2 x_{n-1}, A(t) x_n)$  and set

$$\tilde{G}_n(t, x) = (A(t))^{-(m_0+1)} G_n(t, A(t) * x) = \sum_{|\alpha| \geq m_0} \tilde{A}_\alpha(t) x^\alpha.$$

Then  $\tilde{A}_{(0, \dots, 0, m_0)} \equiv 1$ . If  $A(0) \neq 0$  then  $\tilde{G}_n = G_n$ . In both cases  $\tilde{G}_n$  is regular in  $x_n$ .

*Step 3.* Denote by  $H_n$  the Weierstrass polynomial in  $x_n$  associated to  $\tilde{G}_n$ . It is of degree  $m_0$  in  $x_n$ . Let  $\mathcal{K}$  be the field of fractions of  $\mathcal{O}_{T \times \mathbb{K}^{n-1}, 0}$  and consider  $H_n$  as a polynomial of  $\mathcal{K}[x_n]$ . Let  $d$  be the degree of  $H_{n,\text{red}}$ . We define  $G_{n-1}$  as the  $(d+1)$  generalized discriminant of  $H_n$ , see Appendix II, and set  $Z_n = A^{-1}(0)$ . Let  $F_n$  be the Weierstrass polynomial at  $(t, 0) \in T \setminus (Z_n \cup \text{Sing}(T))$  associated to  $H_n$ . Then, as a germ at  $(t, 0)$ , the discriminant of  $F_{n,\text{red}}$  divides  $G_{n-1}$ .

Then we repeat these steps to  $G_{n-1}$  and so on. This ends the proof of Lemma 6.3 and Theorem 6.2.  $\square$

## 7. STRATIFICATIONS AND WHITNEY FIBERING CONJECTURE.

Let  $X$  be a  $\mathbb{K}$ -analytic space of dimension  $n$ . By an *analytic stratification* of  $X$  we mean a filtration of  $X$  by analytic subspaces

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0$$

such that  $\dim X_j = j$  or  $X_j$  is empty, and  $\text{Sing}(X_j) \subset X_{j-1}$ . This filtration induces a decomposition  $X = \sqcup S_i$ , where the  $S_i$  are connected components of all  $X_j \setminus X_{j-1}$ . The analytic locally closed submanifolds  $S_i$  of  $X$  are called *strata* and their collection  $\mathcal{S} = \{S_i\}$  is usually called a stratification of  $X$ . In what follows we simply say that  $\mathcal{S} = \{S_i\}$  is an analytic stratification of  $X$ , meaning that it comes from an analytic filtration. Similarly we define an algebraic stratification of an algebraic variety.

Stratifications are often considered with extra regularity conditions such as Whitney's (a) and (b) conditions and (w) condition of Verdier. For definitions we refer the reader to [61], [62], [14], and [57]. Recall that for real analytic stratification the (w) condition implies the conditions (a) and (b). For complex analytic stratification the conditions (w) and (b) are equivalent.

We say that a stratification  $\mathcal{S} = \{S_i\}$  is compatible with  $Y \subset X$  if  $Y$  is a union of strata.

**7.1. (Arc-a) and (arc-w) stratifications.** Let  $X$  be a  $\mathbb{K}$ -analytic space and let  $\mathcal{S}$  be an analytic stratification of  $X$ . Let  $p$  be a point of a stratum  $S \in \mathcal{S}$ . We say that  $\mathcal{S}$  is *arc-wise analytically trivial at  $p$*  or, *satisfies condition (arc-a) at  $p$* , if there are a neighborhood  $\Omega$  of  $p$ ,  $\mathbb{K}$ -analytic coordinates on  $\Omega$  such that  $B = S \cap \Omega$  is a neighborhood of the origin in  $\mathbb{K}^m \times \{0\}$ , and an arc-wise analytic trivialization of the projection  $\pi$  on the first  $m$  coordinates

$$(7.1) \quad \Phi(t, x) : B \times \Omega_0 \rightarrow \Omega,$$

where  $\Omega_0 = \Omega \cap \pi^{-1}(0)$ , such that  $\Phi(B \times \{0\}) = B$  and  $\Phi$  preserves the stratification. By the last condition we mean that each stratum of  $\mathcal{S}$  is the union of leaves of  $\Phi$ , see Section 1. We say, moreover, that  $\mathcal{S}$  is regularly *arc-wise analytically trivial at  $p$*  or, *satisfies condition (arc-w) at  $p$* , if  $\Phi$  of (7.1) is regular along  $B \times \{0\}$ .

We say that  $\mathcal{S}$  is *arc-wise analytically trivial*, or *satisfies condition (arc-a)*, if it does it at every point of  $X$ . Similarly we define *regularly arc-wise analytically trivial stratification*. If  $\mathcal{S}$  is regularly arc-wise analytically trivial then, for short, we say that  $\mathcal{S}$  *satisfies (arc-w) condition*.

We say that *the condition (arc-a), resp. (arc-w), is satisfied along a stratum  $S$*  if it is satisfied at every  $p \in S$ . Similarly we say that *the condition (a) or (w) is satisfied along a stratum  $S$*  if for every other stratum  $S'$ ,  $S \subset \overline{S'}$ , the pair  $S', S$  satisfies the respective condition.

**Theorem 7.1.** *If a stratification  $\mathcal{S}$  satisfies (arc-a), resp. (arc-w), condition along a stratum  $S$  then it satisfies the condition (a) of Whitney, resp. the condition (w) of Vedier along  $S$ .*

*Proof.* The first claim follows from the continuity of the tangent space spaces to the leaves of an arc-wise analytic trivialization, c.f. Proposition 1.3.

We show the second claim. Fix two strata  $S_0 \subset \bar{S}_1$ . If the (w) condition fails for the pair  $S_1, S_0$  at  $p_0 \in S_0$ , then, by the curve selection lemma, it fails along a real analytic curve  $p(s) : [0, \varepsilon) \rightarrow \bar{S}_1$  with  $p_0 = p(0) \in S_0$  and  $p(s) \in S_1$  for  $s > 0$ . We show that there is a  $C^1$  submanifold  $(M, \partial M) \subset (\bar{S}_1, S_0)$ ,  $\partial M = S_0$  near  $p_0$ ,  $p(s) \in M$  for  $s$  close to 0, such that, moreover,  $M \setminus \partial M, \partial M$  satisfies the (w) condition. It then follows, see for instance [10], that condition (w) is satisfied along  $p(s)$  for the pair  $S_1, S_0$ , contradicts the choice of  $p(s)$  and hence completes the proof.

We define  $M$  using the trivialization  $\Phi(t, x) = (t, \Psi(x))$  of (7.1). By the arc-analyticity of  $\Phi^{-1}$  there is a real analytic arc  $(t(s), x(s))$  such that  $p(s) = \Phi(t(s), x(s))$ . Then we set

$$(7.2) \quad M = \{\Phi(t, x(s)); t \in B, s \geq 0\}.$$

It follows from Proposition 1.6 that  $M$  is  $C^1$  manifold with boundary and that  $M \setminus \partial M, \partial M$  satisfies the (w) condition.  $\square$

**Corollary 7.2** ([52]). *If a complex analytic hypersurface  $X$  is Zariski equisingular along a non-singular subspace  $Y \subset \text{Sing}(X)$ , and the equisingularity is defined by "generic" projections, then the pair  $\text{Reg}(X), Y$  satisfies Whitney's conditions (a) and (b).*

In the above proof of Theorem 7.1 we use the Wing Lemma argument, the manifold  $M$  being the wing. This method was introduced by Whitney in [61] and then was used by many authors to show the existence of stratifications satisfying various regularity conditions, see for instance [10], [4]. For a wing  $(M, \partial M)$  the condition of being a  $C^1$  submanifold is not sufficient to guarantee the (w) condition for the pair  $M \setminus \partial M, \partial M$ , see [8] for examples. Thus it is essential that our wings admit a parametrization (7.2) satisfying (1.5) of Proposition 1.6. Moreover, we may show the existence of wings that admit a  $\mathbb{K}$ -analytic parametrization and contain a given real analytic arc.

**Proposition 7.3** (Wing Lemma). *Let  $\mathcal{S}$  be an (arc-a) stratification of a  $\mathbb{K}$ -analytic space  $X$ . Let  $p(s) : [0, \varepsilon) \rightarrow X$  be a real analytic arc such that  $p_0 = p(0) \in S_0$  and  $p(s) \in S_1$  for  $s > 0$  and a pair of strata  $S_0, S_1$ . Then, there are a local system of coordinates at  $p_0$ ,  $(X, p_0) \subset (\mathbb{K}^N, 0)$ , an open neighborhood  $\Omega$  of  $p_0$  in  $\mathbb{K}^N$  such that  $B = S_0 \cap \Omega$  is a neighborhood of  $p_0$  in  $\mathbb{K}^m \times \{0\}$ , a neighborhood  $D$  of 0 in  $\mathbb{K}$ , and  $\mathbb{K}$ -analytic maps*

$$t(s) : D \rightarrow B, \quad \varphi : B \times D \rightarrow X, \quad \varphi(t, s) = (t, \psi(t, s)),$$

*such that  $\varphi(t, 0) = (t, 0) \in S_0$ ,  $p(s) = \varphi(t(s), s)$  for  $s > 0$ , and*

- $\varphi(t, s) \in S_1$  for  $s > 0$  if  $\mathbb{K} = \mathbb{R}$
- $\varphi(t, s) \in S_1$  for  $s \neq 0$  if  $\mathbb{K} = \mathbb{C}$ .

*Moreover if  $\mathcal{S}$  is an (arc-w) stratification and if we write  $\psi(t, s) = \sum_{k \geq k_0} D_k(t) s^k$ , then we may require that  $D_{k_0}(0) \neq 0$ .*

*Proof.* The real case follows from the definition of an arc-wise analytic trivialization, and in the regular case from Proposition 1.6. In the complex case we may construct a complex wing as follows. Let  $\Phi(t, x)$  be the arc-wise analytic trivialization given in (7.1) and let  $p(s) = \Phi(t(s), x(s)) : (I, 0) \rightarrow (\bar{S}_1, p_0)$ . Then  $\Phi(t, x(s))$  as a power series defines a complex

analytic map  $\varphi : (T \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^N, 0)$ . Thus  $\varphi(t, s) = \Phi(t, x(s))$  for  $s$  real, but not necessarily for arbitrary  $s \in (\mathbb{C}, 0)$ . Since  $\Phi$  preserves the strata  $\overline{S}_1$  it contains the image of  $\varphi$  for  $s \geq 0$ . Therefore, because  $\overline{S}_1$  is complex analytic near  $p_0$ , it also contains the full image  $\varphi$ .  $\square$

**7.2. Local Isotopy Lemma.** Let  $X$  be a Whitney stratified space,  $p \in X$ , and let  $S$  be the stratum containing  $p$ . Then, as follows from Thoms's first isotopy lemma,  $X$  can be trivialized along  $S$  at  $p$  for any local submersion onto  $S$ . As it follows from Proposition 7.4 below the same phenomenon holds for (arc-a) and (arc-w) stratifications.

Suppose that  $X$  is a  $\mathbb{K}$ -analytic subspace of a neighborhood of the origin in  $\mathbb{K}^N$ ,  $\Omega$  a neighborhood of the origin in  $X$ , and let  $B = \Omega \cap (\mathbb{K}^m \times \{0\})$ . Let  $f$  be a  $\mathbb{K}$ -analytic function on  $X$  and let  $\mathcal{X} = \{X_k\}$  be a finite family of analytic subsets of  $X$ . Let  $\pi : \Omega \rightarrow B$  denote the standard projection onto the first  $m$  coordinates.

**Lemma 7.4.** *Suppose that there exist an arc-wise analytic trivialization  $\Phi$  of  $\pi$  preserving  $B$  and a family of analytic subsets  $\mathcal{X} = \{X_k\}$  of  $X$  and such that  $f$  is regular for  $\Phi$ . Let  $\tilde{\pi}$  be another analytic submersion  $\Omega \rightarrow B$ . Then, after restricting to a smaller neighborhood of the origin, there is an arc-wise analytic trivialization  $\tilde{\Phi}$  of  $\tilde{\pi}$ , preserving  $B$  and the family  $\mathcal{X}$  and such that  $f$  is regular for  $\tilde{\Phi}$ . Moreover, if  $\Phi$  is regular along  $B$  then  $\tilde{\Phi}$  can be chosen regular along  $B$ .*

*Proof.* Let  $H : B \times \Omega_0 \rightarrow B \times \Omega_0$  be given by

$$H(t, x) = (h(t, x), x) = (\tilde{\pi}(\Phi(t, x)), x).$$

We show that  $H$  is a local homeomorphism, arc-wise analytic in  $t$ , such that  $H^{-1}$  is also arc-wise analytic in  $t$ . Then  $\tilde{\Phi} = \Phi \circ H^{-1}$  satisfies the claim.

Firstly  $H$  is a local homeomorphism by the implicit function theorem, Theorem 2.5 of [19]. Let  $\gamma := x(s)$  be a real analytic arc. Consider

$$H_\gamma(t, s) = (h(t, x(s)), s) : (B \times I, 0) \rightarrow (B \times I, 0).$$

Clearly  $H_\gamma$  is  $\mathbb{K}$ -analytic in  $t$  and real analytic in  $s$ . Since  $h(t, x(s)) = t + \varphi(t, s)$  with  $\varphi(t, s) \in \mathfrak{m}_s$ ,  $H_\gamma$  is an analytic diffeomorphism and its inverse is  $\mathbb{K}$ -analytic in  $t$  and real analytic in  $s$ .  $\square$

Let  $\mathcal{S}$  be an analytic stratification of  $X$  satisfying Whitney's condition (a). One says after Définition 4.1.1 of [5] that  $\mathcal{S}$  satisfies stratifying topological triviality condition, (TLS) condition for short, if any local submersion onto a stratum is locally topologically trivial by a preserving strata trivialization. Thus Lemma 7.4 gives the following result. (If we assume that  $\tilde{\pi}$  is only a  $C^1$  submersion then, by Theorem 2.5 of [19],  $H$  constructed in the above proof is a homeomorphism and so is  $\tilde{\Phi}$ .)

**Corollary 7.5.** *A stratification satisfying the (arc-a) condition it also satisfies the condition (TLS) of [5].*

**7.3. Proof of Whitney's fibering conjecture.** We show below that every  $\mathbb{K}$ -analytic space admits locally an (arc-w) stratification. In the algebraic case such a stratification exists globally. Since an (arc-w) stratification satisfies all the properties required by Whitney it shows Whitney fibering conjecture in the algebraic and local analytic cases.

**Theorem 7.6.** *Let  $\mathcal{X} = \{X_i\}$  be a finite family of analytic subsets of an open  $U \subset \mathbb{K}^N$ . Let  $p_0 \in U$ . Then there exists an open neighborhood  $U'$  of  $p_0$  and an analytic stratification of  $U'$  compatible with each  $U' \cap X_i$  and satisfying the condition (arc-w).*

**Theorem 7.7.** *Let  $\mathcal{V} = \{V_i\}$  be a finite family of algebraic subsets of  $\mathbb{P}_{\mathbb{K}}^n$ . Then there exists an algebraic stratification of  $\mathbb{P}_{\mathbb{K}}^n$  compatible with each  $V_i$  and satisfying the condition (arc-w). Moreover, the local arc-wise analytic trivializations can be chosen semi-algebraic.*

Theorem 7.7 will be proven in Section 9. It will follow from Lemma 9.3, the same way as Theorem 7.6 follows from the following version of Lemma 6.3.

**Lemma 7.8.** *Let  $F$  be a  $\mathbb{K}$ -analytic function defined in a neighborhood of  $0 \in \mathbb{K}^N$  and let  $Y$  be a  $\mathbb{K}$ -analytic subset of a neighborhood of  $0 \in \mathbb{K}^N$ ,  $\dim Y = m$ . Then there exist a neighborhood  $U$  of  $0 \in \mathbb{K}^N$  and a  $\mathbb{K}$ -analytic subset  $Z \subset Y \cap U$ ,  $\dim Z < m$ , such that for every  $p \in Y \cap U \setminus Z$  there are a local system of coordinates at  $p$  in which  $(Y, p) = (\mathbb{K}^m \times \{0\}, 0)$  and a Zariski equisingular transverse system pseudopolynomials  $F_i$ ,  $i = 0, \dots, n$ , at  $p$ , such that  $F_n$  is the Weierstrass polynomial associated to  $F$ .*

*Proof.* Choose a local system of coordinates at  $p$  such that the projection on the first  $m$  coordinates restricted to  $Y$  is finite. Let  $n = N - m$  and let

$$f : Y \times \mathbb{K}^n \rightarrow \mathbb{K}^N, \quad f(y, x) = y + (0, x).$$

Then apply Lemma 6.3 to  $T = Y$  and  $F(f(t, x))$ . □

**7.4. Remark on Whitney's fibering conjecture in the complex case.** Let  $U$  be a neighborhood of  $0 \in \mathbb{C}^m \times \mathbb{C}^n$ . Set  $M = U \cap (\mathbb{C}^m \times \{0\})$  and  $N = \{0\} \times \mathbb{C}^n$ . Suppose, following Whitney, that there exists a homeomorphism

$$\phi(p, q) : M \times N \rightarrow U,$$

complex analytic in  $p$ , such that  $\phi(p, 0) = p$  and  $\phi(0, q) = q$ , and that for each  $q \in N$  fixed,  $\phi(\cdot, q) : M \times \{q\} \rightarrow U$  is a complex analytic embedding onto an analytic submanifold  $L(q)$ . Now we make an additional assumption:

(A) *for all  $q \in N$ ,  $L(q)$  is transverse to  $N$ .*

By continuity of  $\phi(p, q)$  we may assume that the projection of  $L(q)$  onto  $M$  is proper. Therefore, by (A) and the assumption  $\phi(0, q) = q$ , it has to be of degree 1. Therefore  $L(q)$  is the graph of a complex analytic function  $f_q : M \rightarrow \mathbb{C}^n$ . If  $q \rightarrow 0$  then the values of  $f_q$  go to 0 and hence, by Cauchy integral formula, the partial derivatives of  $f_q$  go to 0 on relatively compact subsets of  $M$ . This ensures the continuity of the tangent spaces to the leaves  $L(q)$  as  $q \rightarrow 0$ .

**7.5. Examples.** There are several classical examples describing the relation between Zariski equisingularity and Whitney's conditions that we recall below. The general set-up for these examples is the following. Consider a complex algebraic hypersurface  $X \subset \mathbb{C}^4$  defined by a polynomial  $F(x, y, z, t) = 0$  such that  $Sing X = T$ , where  $T$  is the  $t$ -axis. Let  $\pi : \mathbb{C}^4 \rightarrow T$  be the standard projection. In all these examples  $X_t = \pi^{-1}(t)$ ,  $t \in T$ , is a family of isolated singularities, topologically trivial along  $T$ . These examples relate the following conditions

- (1)  $X$  is Zariski equisingular along  $T$ , i.e. there is a local system of coordinates in which  $F$  can be completed to a Zariski equisingular system of polynomials, see Definition 3.1.
- (2)  $X$  is Zariski equisingular along  $T$  for a transverse coordinate system, i.e. there is a local system of coordinates in which  $F$  can be completed to a Zariski equisingular transverse system of polynomials, Section 4.1.
- (3)  $X$  is Zariski equisingular along  $T$  for a generic system of coordinates, i.e. for generic system of local coordinates,  $F$  can be completed to a Zariski equisingular system of polynomials
- (4) The pair  $(X \setminus T, T)$  satisfies Whitney's conditions.

Clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Speder showed (3) $\Rightarrow$ (4) in [52] and (2) $\Rightarrow$ (4) for families of complex analytic hypersurfaces with isolated singularities in  $\mathbb{C}^3$  in his thesis [53] (not published). Theorem 7.1 gives (2) $\Rightarrow$ (4) in the general case.

*Example 7.9 ([6]).*

$$(7.3) \quad F(x, y, z, t) = z^5 + ty^6z + y^7x + x^{15}$$

This example satisfies (1) for the projections  $(x, y, z) \rightarrow (y, z) \rightarrow x$  but (4) fails. As follows from Theorem 7.1, (2) fails as well.

*Example 7.10 ([7]).*

$$(7.4) \quad F(x, y, z, t) = z^3 + tx^4z + y^6 + x^6$$

In this example (4) is satisfied and (3) fails. This example satisfies (1) for the projections  $(x, y, z) \rightarrow (x, z) \rightarrow x$ .

*Example 7.11 ([34]).*

$$(7.5) \quad F(x, y, z, t) = z^{16} + tyz^3x^7 + y^6z^4 + y^{10} + x^{10}$$

In this example (2) is satisfied and (3) fails.

*Example 7.12 ([45]).*

$$(7.6) \quad F(x, y, z, t) = x^9 + y^{12} + z^{15} + tx^3y^4z^5$$

In this example (4) is satisfied and (1) fails.



## 8. EQUISINGULARITY OF FUNCTIONS

In this section we show how to use Zariski equisingularity to obtain local topological triviality of analytic function germs. We develop several different approaches.

Firstly we show that the assumptions of Theorem 3.3 gives not only the topological equisingularity of sets but also of the function  $F_n$  and of any analytic function dividing  $F_n$ . To prove it we modify the vector fields defined by the arc-wise analytic trivialization  $\Phi$ , so that their flows trivialize  $F_n$ . Note that this new trivialization is no longer arc-wise analytic.

Then, for an analytic function  $f$ , we introduce new stratifying conditions (arc- $a_f$ ) and (arc- $w_f$ ), analogs of (arc-a) and (arc-w) conditions, and show that they imply the classical stratifying condition ( $a_f$ ) and ( $w_f$ ).

Finally we show how to adapt Zariski equisingularity to the graph of a function  $f$  in order to obtain an arc-wise analytic triviality of  $f$ .

## 8.1. Zariski equisingularity implies topological triviality of the defining function.

We show that the assumptions of Theorem 3.3 give not only the topological triviality of the zero set of  $F_n$  but also of  $F_n$  as a function.

**Theorem 8.1.** *Let  $B$ ,  $\Omega_0$  and  $\Omega$  be neighborhoods of the origin in  $\mathbb{K}^m$ ,  $\mathbb{K}^n$ , and  $\mathbb{K}^{m+n}$  respectively, and let  $\Phi : B_\varepsilon \times \Omega_0 \rightarrow \Omega$  be an arc-wise analytic trivialization satisfying condition (Z1) of Theorem 3.3. Let  $f(t, x)$  be a  $\mathbb{K}$ -analytic function regular for  $\Phi$ . Then  $f$  is topologically trivial along  $B \times \{0\}$  at the origin, i.e. there are smaller neighborhoods  $B'$ ,  $\Omega'_0$  and  $\Omega'$  and a homeomorphism*

$$h : B' \times \Omega'_0 \rightarrow \Omega'$$

*such that  $h(t, 0) = (t, 0)$ ,  $h(0, x) = (0, x)$ , and  $f(h(t, x)) = f(0, x)$ .*

*Proof.* The trivialization  $h$  is obtained by integrating the vector fields  $w_i(t, x)$  defined below. Let  $v_i$  be the vector fields on  $\Omega$  that defines the trivialization  $\Phi$  that is the vector fields given by (1.2). The regularity condition (1.8) gives

$$(8.1) \quad |\partial f / \partial v_i| \leq C|f|.$$

Note that locally on  $\Omega \setminus V(f)$  we may approximate  $v_i$  by a smooth vector field satisfying (8.1). Using partition of unity, we may glue such local approximations to a smooth vector field that satisfies (8.1) and extends continuously to  $V(f)$  by  $v_i|_{V(f)}$ . In what follows we replace  $v_i$  by such an approximation.

Next we consider on  $\Omega \setminus V(f)$  the orthogonal projection of  $v_i(t, x)$  on the levels of  $f$

$$w_i(t, x) = v_i(t, x) - \frac{\partial f / \partial v_i}{\|\text{grad } f\|^2} \text{grad } f.$$

(in the complex case  $\text{grad } f := (\partial f / \partial z_1, \dots, \partial f / \partial z_{m+n})$  so that  $\partial f / \partial v = \langle v, \text{grad } f \rangle$ ). Then we extend  $w_i$  by  $v_i(t, x)$  onto  $\Omega$ . By the Łojasiewicz Gradient Inequality there are a neighborhood of the origin  $U$  and constants  $C > 0$ ,  $\theta < 1$ , such that

$$\|\text{grad } f\| \geq C|f|^\theta.$$

By (8.1)  $w_i$  are continuous and clearly satisfy  $\partial f / \partial w_i = 0$ . By Remark 1.4 the integral curves of  $w_i|_{V(f)}$  are unique and hence they are unique on  $\Omega$ . Therefore, by Theorem 2.1 of [19], for each  $i$  the flow  $h_i$  of  $w_i$  is continuous. Then we trivialize  $f$  by composing these flows:

$$h(t_1, \dots, t_m, x) = h_1(t_1, h_2(t_2, h_3(\dots(t_{m-1}, h_m(t_m, x)))).$$

□

**8.2. Conditions (arc- $a_f$ ) and (arc- $w_f$ ).** Let  $f : X \rightarrow \mathbb{K}$  be a  $\mathbb{K}$ -analytic function defined on a  $\mathbb{K}$ -analytic space  $X$ . By a *stratification* of  $f$  we mean an analytic stratification  $\mathcal{S}$  of  $X$  such that  $V(f)$  is a union of strata. We also assume that for any stratum  $S \subset X \setminus V(f)$ ,  $f|_S$  has no critical points. A stratification  $\mathcal{S}$  of  $f$  is called a *Thom stratification* of  $f$  if it is a Whitney stratification of  $X$  that for each pair of strata satisfies Thom condition  $(a_f)$ . For a definition of condition  $(a_f)$  we refer the reader to [14], [31]. For the strict Thom condition  $(w_f)$  see [31] and [20].

We say that a stratification  $\mathcal{S}$  of  $f$  *satisfies condition (arc- $a_f$ ) at  $p \in V(f)$*  if there exists a local arc-wise analytic trivialization (7.1) at  $p$  preserving the strata of  $\mathcal{S}$  and such that  $f$  is regular for  $\Phi$  at  $p$ . If, moreover,  $\Phi$  is regular at  $p$  then we say that  $\mathcal{S}$  *satisfies condition (arc- $w_f$ ) at  $p$* .

We say that *the condition (arc- $a_f$ ), resp. (arc- $w_f$ ), is satisfied along a stratum  $S$*  if it is satisfied at every  $p \in S$ . Similarly we say that *the condition  $(a_f)$  or  $(w_f)$  is satisfied along a stratum  $S$*  if for every other stratum  $S'$ ,  $S \subset \overline{S'}$ , the pair  $S', S$  satisfies the respective condition.

We note that by the assumption that  $f|_S$  has no critical points on stratum  $S \subset X \setminus V(f)$ , the levels of  $f$  are transverse to  $S$ . Therefore, if moreover  $\mathcal{S}$  satisfies Whitney's condition (a), the conditions  $(a_f)$  and  $(w_f)$  are automatically satisfied along  $S$ .

**Theorem 8.2.** *If a stratification of  $f$  satisfies the (arc- $a_f$ ), resp. (arc- $w_f$ ), condition along a stratum  $S \subset V(f)$  then it satisfies the Thom condition  $(a_f)$ , resp.  $(w_f)$ , condition along  $S$ .*

*Proof.* Similarly to the proof of theorem 7.1 it suffices we check the conditions along a real analytic curve by considering a wing containing the curve.

Thus fix two strata  $S_0 \subset \overline{S_1}$ ,  $S_0 \subset V(f)$ ,  $S_1 \cap V(f) = \emptyset$ , and a real analytic curve  $\gamma : p(s) = \Phi(t(s), x(s)) : [0, \varepsilon) \rightarrow \overline{S_1}$  with  $p_0 = p(0) \in S_0$  and  $p(s) \in S_1$  for  $s > 0$ . First we consider the case  $\mathbb{K} = \mathbb{R}$  and the wing

$$M = \{\Phi(t, x(s)); t \in B, s \geq 0\}.$$

By the regularity of  $f$  for  $\Phi$ , Proposition 1.7, we may reparametrize  $\Phi_\gamma(t, s) = \Phi(t, x(s))$  by replacing  $s = s(t, \tilde{s})$  so that  $f(\Phi_\gamma(t, \tilde{s})) = \tilde{s}^{k_0}$ . If we write  $\Phi_\gamma(t, \tilde{s}) = (t, \Psi_\gamma(t, \tilde{s}))$  then the tangent space to the levels of  $f|_M$  is generated by  $D\Phi_\gamma(\partial/\partial t_i, \partial\Psi_\gamma/\partial t_i)$ , that tends to  $(\partial/\partial t_i, 0)$  as  $s \rightarrow 0$ ,  $i = 1, \dots, m$ . This shows  $(a_f)$ . If moreover  $\Phi$  is regular then the condition  $(w_f)$  follows from Proposition 1.6.

If  $\mathbb{K} = \mathbb{C}$  then we use the complex wing of Proposition 7.3 and proceed as in the real case.  $\square$

**Corollary 8.3.** *Let  $f : X \rightarrow \mathbb{K}$  be  $\mathbb{K}$ -analytic and let  $\mathcal{S}$  be a Whitney stratification of  $f$  satisfying the condition (arc- $a_f$ ). Then  $f$  is topologically trivial along each stratum  $S \subset V(f)$ .*

In the complex analytic case it is shown in [5] that any stratification of  $f$  satisfying the conditions (a) and (TLS), it also satisfies the condition ( $a_f$ ). Similarly, after [5] and [47] any Whitney stratification of  $f$  satisfies the strong Thom condition ( $w_f$ ). Analogous results are false in the real analytic case. Thus in the complex case Theorem 8.2 (for the whole stratification and not for a single stratum) follows from Theorem 7.1, Proposition 1.13, and Corollary 7.5.

Thom's condition ( $a_f$ ) for a function  $f$  ensures its topological triviality along strata of a Whitney stratification. But the condition (arc- $a_f$ ) itself does not imply Whitney's condition (b) and therefore it does not follow from the stratification theory that it implies topological triviality of  $f$  along strata. In some special case this can be obtained by adapting the proof of Theorem 8.1.

**Corollary 8.4.** *Let  $f : X \rightarrow \mathbb{K}$  be  $\mathbb{K}$ -analytic and let  $\mathcal{S}$  be a stratification of  $f$  such that  $X \setminus V(f)$  is a stratum of  $\mathcal{S}$ . Suppose that  $\mathcal{S}$  satisfies the condition (arc- $a_f$ ) along a stratum  $S \subset V(f)$ . Then  $f$  is topologically trivial along  $S$ .*

**8.3. Arc-wise analytic triviality of function germs.** Consider a family of function germs  $f_t(y) = f(t, y) : T \times (\mathbb{K}^{n-1}, 0) \rightarrow \mathbb{K}$ , parametrized by an open  $T \subset \mathbb{K}^m$ . We say that  $f_t$  is *arc-wise analytically trivial along  $T$*  if there are neighborhoods  $\Lambda$  of  $T \times \{0\}$  in  $\mathbb{K}^m \times \mathbb{K}^{n-1}$  and  $\Lambda_0$  of  $\{0\}$  in  $\mathbb{K}^{n-1}$ ,  $f_0 : \Lambda_0 \rightarrow \mathbb{K}$ , and an arc-wise analytic analytic trivialization

$$\sigma : T \times \Lambda_0 \rightarrow \Lambda, \text{ such that } f(\sigma(t, y)) = f_0(y).$$

Using the method developed in [3] we have the following result.

**Theorem 8.5.** *Let  $f_t(y) = f(t, y) : T \times (\mathbb{K}^{n-1}, 0) \rightarrow \mathbb{K}$  be a  $\mathbb{K}$ -analytic family of  $\mathbb{K}$ -analytic function germs and let  $t_0 \in T$ . Then there exist a neighborhood  $U$  of  $t_0$  in  $T$  and a  $\mathbb{K}$ -analytic subset  $Z \subset U$  such that  $f$  is arc-wise analytically trivial along  $U \setminus Z$ .*

*Proof.* Set  $x = (x_1, \dots, x_n) = (x_1, y)$ , where  $y = (y_1, \dots, y_{n-1})$ , and  $F(t, x_1, y) = x_1 - f(t, y)$ . Then apply to  $F$  Lemma 6.3 and Theorem 3.3, together with Proposition 3.10, in order to get an arc-wise analytic trivialization  $\Phi$  of the zero set of  $F$  that preserves the levels of  $x_1$ . Then

$$\sigma(t, y) = \pi(\Phi(t, f_0(y), y)),$$

where  $\pi$  denotes the projection  $\pi(t, x_1, y) = (t, y)$ , gives an arc-wise analytic trivialization of  $f$ .  $\square$

## 9. ALGEBRAIC CASE

In the algebraic case we have a global version of Theorem 6.2. Here by a *real algebraic variety* we mean an affine real algebraic variety in the sense of Bochnak-Coste-Roy [4]:

a topological space with a sheaf of real-valued functions isomorphic to a real algebraic set  $X \subset \mathbb{R}^N$  with the Zariski topology and the structure sheaf of regular functions. For instance, the set of real points of a reduced projective scheme over  $\mathbb{R}$ , with the sheaf of regular functions, is an affine real algebraic variety in this sense.

**Theorem 9.1.** *Let  $T$  be an algebraic variety (over  $\mathbb{K}$ ) and let  $\mathcal{X} = \{X_k\}$  be a finite family of algebraic subsets  $T \times \mathbb{P}_{\mathbb{K}}^{n-1}$ . Then there exists an algebraic stratification  $\mathcal{S}$  of  $T$  such that for every stratum  $S$  and for every  $t_0 \in S$  there is a neighborhood  $U$  of  $t_0$  in  $S$  and a semialgebraic arc-wise analytic trivialization of  $\pi$ , preserving the family  $\mathcal{X}$ ,*

$$(9.1) \quad \Phi : U \times \mathbb{P}_{\mathbb{K}}^{n-1} \rightarrow \pi^{-1}(U),$$

$\Phi(t_0, x) = (t_0, x)$ , where  $\pi : T \times \mathbb{P}_{\mathbb{K}}^{n-1} \rightarrow T$  denotes the projection.

*Proof.* We may assume that  $T$  is affine irreducible. Let  $G_i(t, x)$ ,  $t \in T$ ,  $x = (x_1, \dots, x_n)$ , be a finite family of polynomials, homogeneous in  $x$ , defining the subsets of family  $\mathcal{X}$  and let  $F_n(t, x)$  be the product of all  $G_i$ s. We consider  $F_n$  as a homogeneous polynomial over  $\mathcal{K} = \mathbb{K}(T)$  and let

$$F_n(x) = \sum_{|\alpha|=d_n} A_\alpha x^\alpha, \quad A_\alpha \in \mathcal{K}.$$

After a linear change of coordinates  $x$ , we may suppose  $A_n = A_{(0, \dots, 0, d_n)} \neq 0$ . Then we define  $F_{n-1}(x_1, \dots, x_{n-1})$  as the discriminant of  $F_{n, red}$ , and proceed inductively by constructing the system of homogeneous polynomials  $F_i \in \mathcal{K}[x_1, \dots, x_i]$  until  $F_i = 1$ . Then we take as  $Z \subset T$  the union of zero sets of the denominators of coefficients of all  $F_j$ . We show below that the statement of theorem holds for  $T \setminus Z$  as an open stratum. Then the stratification  $\mathcal{S}$  can be constructed by induction on  $\dim T$ .

By Theorem 3.3,  $V(G)$  is arc-wise analytically equisingular along  $(T \setminus Z) \times \{0\}$ . By construction (3.6), the trivialisation  $\Phi(t, x) = (t, \Psi(t, x))$  is semi-algebraic and  $\mathbb{K}^*$ -equivariant in variable  $x$ , as follows from the interpolation formula, see Remark 9.7. Moreover by construction it is regular along  $U \times \{0\}$ . Then the induced trivialization (denoted also by  $\Phi$ ),  $\Phi : U \times \mathbb{P}_{\mathbb{K}}^{n-1} \rightarrow \pi^{-1}(U)$  is arc-wise analytic. Indeed, let  $x(s) : I \rightarrow \mathbb{P}_{\mathbb{K}}^{n-1}$  be a real analytic arc that, after a linear change of coordinates, we may assume to be of the form

$$x(s) = [x_1(s) : \dots : x_{n-1}(s) : 1],$$

with  $|x_i(0)|$ , for  $i = 1, \dots, n-1$ , bounded by a universal constant. If we write  $\Psi = [\Psi_1 : \dots : \Psi_n]$ , then  $\Psi_n(t, x(s))$  does not vanish and we may replace  $\Psi(x(s))$ , by  $[\Psi_1/\Psi_n x(s) : \dots : \Psi_{n-1}/\Psi_n x(s) : 1]$ . It follows from the regularity of  $\Phi$  along  $U \times \{0\}$  that  $\Psi_n(t, x(s)) \neq 0$  in a fixed neighborhood of  $p$  in  $S$  independent of  $x(s)$ . This shows the claim.  $\square$

**9.1. Proof of Theorem 7.7.** We have the following algebraic versions of Lemmas 6.3 and 7.8.

**Lemma 9.2.** *Let  $T$  be a  $\mathbb{K}$  algebraic variety and let  $F \in \mathbb{K}[T \times \mathbb{K}^n]$ ,  $F \neq 0$ . Then there exists a subvariety  $Z \subset T$ ,  $\dim Z < \dim T$ , such that, after a linear change of coordinates*

in  $\mathbb{K}^n$ ,  $F$  can be completed to a system of polynomials  $\{F_i\}$ ,  $F_n = F$ , such that for every  $t \in T \setminus Z$  the system  $\{F_i\}$  is transverse and Zariski equisingular at  $(t, 0)$ .  $\square$

**Lemma 9.3.** *Let  $F \in \mathbb{K}[X_1, \dots, X_N]$ ,  $F \not\equiv 0$ , and let  $Y \subset \mathbb{K}^N$  be an algebraic subset. Then there exist an algebraic  $Z \subset Y$ ,  $\dim Z < \dim Y$ , and polynomials  $\{F_i\}$ ,  $F_n = F$ , such that the following holds. For every  $p \in Y \setminus Z$  there is a local system of coordinates at  $p$  in which  $(Y, p) = (\mathbb{K}^m \times \{0\}, 0)$ , such that the germs of  $\{F_i\}$  at  $p$  form a transverse and Zariski equisingular system.*  $\square$

Lemma 9.3 implies the algebraic Whitney's fibering conjecture: Theorem 7.7

**9.2. Applications to real algebraic geometry.** Let  $X$  be a compact (projective or affine) real algebraic variety in the sense of [4]. A functorial filtration on the semi-algebraic chains  $C_*(X; \mathbb{Z}_2)$  was introduced in [36]. This filtration, called *the Nash filtration*, defines a spectral sequence, *the weight spectral sequence of  $X$* , that, in turn, defines *the weight filtration* on the homology  $H_*(X; \mathbb{Z}_2)$ . This construction can be extended to non-compact real algebraic varieties and the Borel-Moore homology. For a real algebraic variety  $X$  its virtual Poincaré polynomial  $\beta(X) \in \mathbb{Z}[t]$ , introduced in [35], is a multiplicative and additive invariant, an analog of the Hodge Deligne polynomial. As shown in [36], the virtual Poincaré polynomial can be computed from the weight spectral sequence. For the cohomological counterpart of this theory see [32].

The Nash filtration is functorial not only for regular morphisms but also for the  $\mathcal{AS}$ -maps that can be defined as follows. Let  $X, Y$  be compact real algebraic varieties. A continuous map  $f : X \rightarrow Y$  is an  $\mathcal{AS}$ -map if its graph  $\Gamma_f$  is a semialgebraic and arc-symmetric subset of  $X \times Y$ . For instance a map that is semialgebraic and arc-analytic is  $\mathcal{AS}$ . For more on  $\mathcal{AS}$  maps see [49], [30].

Let  $\Phi$  be a semialgebraic arc-wise analytic trivialization (9.1) preserving real algebraic  $X \subset T \times \mathbb{P}_{\mathbb{K}}^{n-1}$  and let  $X_t = \pi^{-1}(t)$ . Then for each  $t \in U$ ,  $\Phi$  induces a semialgebraic and arc-analytic homeomorphism

$$\varphi_{t_0, t} : X_{t_0} \rightarrow X_t,$$

with an arc-analytic inverse. In particular, each  $\varphi_{t_0, t}$  is  $\mathcal{AS}$ . Thus Theorem 9.1 gives the following.

**Corollary 9.4.** *Let  $T$  be a real algebraic variety and let  $X$  be an algebraic subset of  $T \times \mathbb{P}_{\mathbb{K}}^{n-1}$ . Then there exists a finite stratification  $\mathcal{S}$  of  $T$  such that for every stratum  $S$  and for every  $t_0, t_1 \in S$  the fibres  $X_{t_0}$  and  $X_{t_1}$  are  $\mathcal{AS}$ -homeomorphic and hence have isomorphic weight spectral sequences.*  $\square$

**Corollary 9.5.** *Let  $T$  be a real algebraic variety and let  $X$  be an algebraic subset of  $T \times \mathbb{P}_{\mathbb{K}}^{n-1}$ . Then there exists a finite stratification  $\mathcal{S}$  of  $T$  such that for every stratum  $S$  the virtual Poincaré polynomial  $\beta(X_t)$  is independent of  $t \in S$ .*  $\square$

The latter result was also shown in [11] by means of the resolution of singularities.

## Appendix I. Whitney Interpolation.

We generalize the classical Whitney Interpolation formula [62], [18].

Fix positive integers  $d, N$  and consider a family of functions  $f_i : \mathbb{C}^N \rightarrow \mathbb{C}$ ,  $i = 1, 2, \dots, N$ . We assume that, for a constant  $C > 1$ , this family satisfies the following properties

- (1)  $f_i$  are continuous, differentiable on  $(\mathbb{C}^*)^N$ , and  $\mathbb{R}$ -homogeneous of degree  $d$ .
- (2) for every permutation  $\sigma \in S_N$ :  $f_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(N)}) = f_{\sigma(i)}(\xi_1, \dots, \xi_N)$ .
- (3)  $|f_j(\xi_1, \dots, \xi_N)| \leq C|\xi_j|(\max_i |\xi_i|)^{d-1}$ .
- (4) for all  $k, j$ ,  $|\xi_k^2| \|(\partial f_j / \partial \xi_k, \partial f_j / \partial \xi_k)\| \leq C|\xi_j|(\max_i |\xi_i|)^d$
- (5)  $f = \sum_i f_i$  is real valued and satisfies  $C^{-1}(\max_i |\xi_i|)^d \leq f(\xi_1, \dots, \xi_N) \leq C(\max_i |\xi_i|)^d$ .

For examples see Examples 9.11 and 9.12.

Given two subsets  $\{a_1, \dots, a_N\} \subset \mathbb{C}$ ,  $\{b_1, \dots, b_N\} \subset \mathbb{C}$ , of cardinality  $N$  such that if  $a_i = a_j$  then  $b_i = b_j$ . Define  $D_i = b_i - a_i$  and set

$$(9.2) \quad \gamma = \max_{a_i \neq a_j} \frac{|D_i - D_j|}{|a_i - a_j|}.$$

Then

$$(9.3) \quad |D_i - D_j| \leq \gamma |a_i - a_j|.$$

Let

$$\mu_i(z) := f_i((z - a_1)^{-1}, \dots, (z - a_N)^{-1}), \quad \mu(z) := f((z - a_1)^{-1}, \dots, (z - a_N)^{-1}).$$

Define the interpolation map  $\psi : \mathbb{C} \rightarrow \mathbb{C}$  by

$$(9.4) \quad \psi(z) = z + \frac{\sum_{i=1}^N \mu_i(z) D_i}{\mu(z)},$$

if  $z \notin \{a_1, \dots, a_N\}$  and  $\psi(a_i) = b_i$ . Then  $\psi$  is continuous as follows from the following lemma.

**Lemma 9.6.**

$$\lim_{z \rightarrow a_j} \psi(z) = b_j.$$

*Proof.* Let  $I_j = \{i; a_i = a_j\}$ . We rewrite the interpolation formula (9.4) as

$$(9.5) \quad \psi(z) = z + D_j + \frac{\sum_{i \notin I_j} \mu_i(z) (D_i - D_j)}{\mu(z)}.$$

By the properties (3) and (5), for  $i \notin I_j$ ,  $\frac{\mu_i(z)}{\mu(z)} \rightarrow 0$  as  $z \rightarrow a_j$ . □

**Remark 9.7. Symmetries.**

The map  $\psi$  is also invariant under permutations  $\sigma \in S_N$ ,  $\sigma(a) = (a_{\sigma(1)}, \dots, a_{\sigma(N)})$

$$\psi(z, \sigma(a), \sigma(b)) = \psi(z, a, b).$$

Let  $\tau : \mathbb{C} \rightarrow \mathbb{C}$  be complex affine. Then

$$\psi(\tau(z), \tau(a), \tau(b)) = \tau(\psi(z, a, b)).$$

**Proposition 9.8.** *The map  $\psi : \mathbb{C} \rightarrow \mathbb{C}$  is Lipschitz with Lipschitz constant  $4N^2C^4\gamma + 1$ . If  $\gamma < (4N^2C^4)^{-1}$  then  $\psi$  is a bi-Lipschitz homeomorphism, with  $(1 - 4N^2C^4\gamma)^{-1}$  a Lipschitz constant of  $\psi^{-1}$ .*

*Proof.* It suffices to show that for  $z \notin \{a_1, \dots, a_N\}$  and for every unit vector  $v \in \mathbb{C}$

$$(9.6) \quad |(\psi(z) - z)'| \leq 4N^2C^4\gamma,$$

where by “prime” we denote any directional derivative  $\frac{\partial}{\partial v}$ ,  $v \in \mathbb{C}$ ,  $|v| = 1$ . Indeed, if (9.6) holds then clearly  $\psi$  is Lipschitz. Moreover, if  $\gamma < (4N^2C^4)^{-1}$  then for any  $p \in \mathbb{C}$ ,  $z \rightarrow p + z - \psi(z)$  is a contraction and hence admits a unique fixed point  $z_p$ , that is a unique  $z_p$  such that  $\psi(z_p) = p$ . Hence  $\psi$  is a homeomorphism by the invariance of domain. By (9.6) for any  $p, q \in \mathbb{C}$ ,  $|(\psi(p) - p) - (\psi(q) - q)| \leq 4N^2C^4\gamma|p - q|$ , that gives

$$(9.7) \quad |p - q| \leq (1 - 4N^2C^4\gamma)^{-1}|\psi(p) - \psi(q)|$$

if  $\gamma < (4N^2C^4)^{-1}$ .

To show (9.6) we use the following bounds that follow from the conditions (3)-(5).

$$(9.8) \quad \begin{aligned} |\mu_i(z)| &\leq C^2|z - a_i|^{-1}\mu(z)^{\frac{d-1}{d}}, \\ |\mu'_i(z)| &\leq NC^2|z - a_i|^{-1}\mu(z), \\ |\mu'(z)| &\leq N^2C^2\mu(z)^{\frac{d+1}{d}}, \end{aligned}$$

Given  $z \in \mathbb{C}$ , choose  $j$  such that  $|z - a_j| = \min_i |z - a_i|$ . Then, for all  $i$ ,

$$(9.9) \quad |a_i - a_j| \leq 2|z - a_i|.$$

By differentiating (9.5)

$$(9.10) \quad |(\psi(z) - z)'| \leq \frac{\sum_{i \notin I_j} |\mu'_i(z)(D_i - D_j)|}{\mu(z)} + \frac{(\sum_{i \notin I_j} |\mu_i(z)(D_i - D_j)|)|\mu'(z)|}{(\mu(z))^2}.$$

By (9.3) and (9.8)

$$|\mu'_i(z)(D_i - D_j)| \leq 2NC^2\gamma\mu(z)$$

and

$$|\mu_i(z)(D_i - D_j)||\mu'(z)| \leq 2N^2C^4\gamma(\mu(z))^2.$$

This shows (9.6) and hence ends the proof of Proposition 9.8.  $\square$

Consider  $\psi$  as a function of three arguments  $z \in \mathbb{C}$ ,  $a \in \mathbb{C}^N$ ,  $b \in \mathbb{C}^N$ ,

$$(9.11) \quad \psi(z, a, b) = \psi_{a,b}(z) = z + \frac{\sum_{j=1}^N \mu_j(z, a)(b_j - a_j)}{\mu(z, a)},$$

where  $\mu_i(z, a) = f_i((z - a_1)^{-1}, \dots, (z - a_N)^{-1})$ ,  $\mu(z, a) = \sum_i \mu_i(z, a)$ , and  $\psi_{a,b}(a_i) = b_i$ . Then  $\psi(z, a, b)$  can also be considered as a family of functions  $\psi_{a,b} : \mathbb{C} \rightarrow \mathbb{C}$ , parameterized by  $a, b$ .

**Proposition 9.9.** *Let  $a(x) : X \rightarrow \mathbb{C}^N$ ,  $b(x) : X \rightarrow \mathbb{C}^N$  be continuous functions defined on a topological space  $X$  such that for every  $x \in X$  and  $i, j$ , if  $a_i(x) = a_j(x)$  then  $b_i(x) = b_j(x)$ . Then  $\psi(z, a(x), b(x))$  is continuous as a function of  $(x, z)$ .*

*Proof.* Let  $(z, a, b) \rightarrow (z_0, a_0, b_0)$ . Clearly  $\psi(z, a, b) \rightarrow \psi(z_0, a_0, b_0)$  if  $z_0 \notin \{a_{01}, \dots, a_{0N}\}$ . Thus suppose  $z_0 = a_{01} = a_{0j}$  for  $j \in J$ . Then  $\psi(z_0, a_0, b_0) = b_{01}$ . We have

$$\begin{aligned} \psi(z, a, b) - \psi(z_0, a_0, b_0) &= (z - z_0) + \frac{\sum_{i \in J} \mu_i(z, a)((b_i - b_{0i}) - (a_i - a_{0i}))}{\mu(z, a)} \\ &\quad + \frac{\sum_{i \notin J} \mu_i(z, a)((b_{0i} - b_{01}) - (a_{0i} - a_{01}))}{\mu(z, a)}. \end{aligned}$$

Clearly the first two summands converge to 0 as  $(z, a, b) \rightarrow (z_0, a_0, b_0)$ . So does the third one because

$$\frac{\mu_i(z, a)}{\mu(z, a)} \rightarrow 0$$

if  $i \notin J$ . To show this last property we note that  $\mu(z, a) \rightarrow \infty$ , that the limit of  $z - a_i$  is nonzero if  $i \notin J$ , and use the first inequality of (9.8).  $\square$

*Remark 9.10.* If  $(a, b) \rightarrow (a_0, b_0)$  then  $\gamma(a_0, b_0) \leq \liminf \gamma(a, b)$ , thus  $\gamma$  is lower semi-continuous.

*Example 9.11.* In the original Whitney interpolation  $f_i(\xi) = |\xi_i|$ , cf. [62], see also [18].

*Example 9.12.* In this paper we use the following family. For  $\xi_1, \dots, \xi_N \in \mathbb{C}$  we denote by  $\sigma_i = \sigma_i(\xi_1, \dots, \xi_N)$  the elementary symmetric functions of  $\xi_1, \dots, \xi_N$ . Let  $P_k = \sigma_k^{\alpha_k}$ , where  $\alpha_k = (N!)/k$ . Define

$$(9.12) \quad f(\xi) = \sum_k P_k(\xi) \bar{P}_k(\xi).$$

and

$$(9.13) \quad f_j(\xi) = \frac{1}{N!} \sum_k \xi_j \frac{\partial P_k}{\partial \xi_j} \bar{P}_k.$$

Then  $\psi$  equals

$$(9.14) \quad \psi(z, a, b) = z + \frac{\sum_k (\sum_{j=1}^N \xi_j \frac{\partial P_k}{\partial \xi_j} (b_j - a_j)) \bar{P}_k(\xi)}{N! (\sum_k P_k \bar{P}_k(\xi))},$$

where  $\xi = ((z - a_1)^{-1}, \dots, (z - a_N)^{-1})$ .



## Appendix II. Generalized discriminants.

Let  $\mathcal{K}$  be a field of characteristic zero and let

$$F(Z) = Z^p + \sum_{j=1}^p a_j Z^{p-j} = \prod_{j=1}^p (Z - \xi_j) \in \mathcal{K}[Z],$$

with the roots  $\xi_i \in \overline{\mathcal{K}}$ . Then the expressions

$$\Delta_j = \sum_{r_1, \dots, r_{j-1}} \prod_{k < l; k, l \neq r_1, \dots, r_{j-1}} (\xi_k - \xi_l)^2$$

are symmetric in  $\xi_1, \dots, \xi_p$  and hence polynomials in  $a_1, \dots, a_p$ . Thus  $\Delta_1$  is the standard discriminant and  $F$  has exactly  $p - j$  distinct roots if and only if  $\Delta_1 = \dots = \Delta_j = 0$  and  $\Delta_{j+1} \neq 0$ . The following lemma is obvious.

**Lemma 9.13.** *Let  $F \in \mathcal{K}[Z]$  be a monic polynomial of degree  $p$  that has exactly  $d$  distinct roots in  $\xi_i \in \overline{\mathcal{K}}$ . Then the generalized discriminant  $\Delta_{d+1, F}$  of  $F$  and the standard discriminant  $\Delta_{F_{red}}$  of  $F_{red}$  are related by*

$$\Delta_{d+1, F} = \binom{p}{d} \Delta_{F_{red}}.$$

We use often the following consequence of the Implicit Function Theorem.

**Lemma 9.14.** *Let  $F \in \mathbb{K}\{x_1, \dots, x_n\}[Z]$  be a monic polynomial in  $Z$  such that the discriminant  $\Delta_{F_{red}}$  does not vanish at the origin. Then, on a neighborhood  $U$  of  $0 \in \mathbb{K}^n$ , the complex roots  $\xi_i(x_1, \dots, x_n)$  of  $F$  are  $\mathbb{K}$ -analytic, distinct, and of constant multiplicities.*

## REFERENCES

- [1] S. S. Abhyankar, On the ramification of algebraic functions, *Amer. J. Math.*, **77**, (1955), 575-592.
- [2] E. Bierstone, P. D. Milman, Arc-analytic functions, *Invent. math.* **101** (1990), 411-424.
- [3] M. Bilski, A. Parusiński, G. Rond, Local topological algebraicity of analytic function germs. *J. Algebraic Geom.*, to appear
- [4] J. Bochnak, M. Coste, M.-F. Roy, *Real Algebraic Geometry*, Springer-Verlag, Berlin Heidelberg 1998.
- [5] J. Briançon, Ph. Maisonobe, M. Merle, Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom. (French) *Invent. Math.* **117** (1994), no. 3, 531-550.
- [6] J. Briançon, J.-P. Speder, La trivialité topologique n'implique pas les conditions de Whitney. *C. R. Acad. Sci. Paris Sér. A-B* **280** (1975), no. 6, A365-A367.
- [7] J. Briançon, J.-P. Speder, Familles équisingulières de surfaces  $\tilde{A}$  singularité isolée. *C. R. Acad. Sci. Paris Sér. A-B* **280** (1975), Aii, A1013-A1016.
- [8] H. Brodersen, D. Trotman, Whitney (b)-regularity is weaker than Kuo's ratio test for real algebraic stratifications. *Math. Scand.* **45** (1979), no. 1, 27-34.
- [9] F. Bruhat, H. Cartan, Sur la structure des sous-ensembles analytiques réels, *C. R. Acad. Sci.* **244** (1957), 988-990.
- [10] Z. Denkowska, K. Wachta, Une construction de la stratification sous-analytique avec la condition (w), *Bull. Polish Acad. Sci. Math.* **35** (1987), no. 7-8, 401-405
- [11] G. Fichou, M. Shiota Virtual Poincaré polynomial of the link of a real algebraic variety. *Math. Z.* **273** (2013), no. 3-4, 1053-1061.

- [12] G. Fichou, J. Huisman, F. Mangolte, and J.-Ph. Monnier, Fonctions régulières, arXiv: 1112.3800v3 [math.AG], to appear in *J. Reine Angew. Math.*
- [13] T. Fukui, Seeking invariants for blow-analytic equivalence, *Compositio Math.* **105** (1997), 95–107.
- [14] C. G. Gibson, K. Wirthmüller, A. A. du Plessis and E. J. N. Looijenga, Topological stability of smooth mappings, *Lecture Notes in Math.* **552**, Springer-Verlag, 1976.
- [15] R. Hardt, Topological properties of subanalytic sets. *Trans. Amer. Math. Soc.* **211** (1975), 57–70.
- [16] R. Hardt, Semi-algebraic local-triviality in semi-algebraic mappings. *Amer. J. Math.* **102** (1980), no. 2, 291–302.
- [17] R. Hardt, Stratification via corank 1 projections, in *Singularities, Proc. Symp. in Pure Math.*, **40** (1983), 559–566.
- [18] R. Hardt, D. Sullivan, Variation of the Green function on Riemann surfaces and Whitney’s holomorphic stratification conjecture. *Inst. Hautes Études Sci. Publ. Math.* No. **68** (1988), 115–137.
- [19] P. Hartman, *Ordinary Differential Equations*, (Wiley, 1964).
- [20] J. P. G. Henry, M. Merle, C. Sabbah, Sur la condition de Thorn stricte pour un morphisme analytique complexe. *Annales E.N.S.*, **17**, 2 (1984), p. 227–268.
- [21] H. Hironaka, Subanalytic sets, *Number Theory, Algebraic Geometry and Commutative Algebra*, Volume in honour of Y. Akizuki, pp. 453–493. Kinokuniya, Tokyo, 1973.
- [22] S. Koike, Finiteness theorems on blow-Nash triviality for real algebraic singularities. *Geometric singularity theory*, 135–149, Banach Center Publ., 65, Polish Acad. Sci., Warsaw, 2004.
- [23] J. Kollár, K. Nowak, Continuous Rational Functions on Real and p-adic Varieties, arXiv: 1301.5048v2 [math.AG], to appear in *Math. Z.*
- [24] W. Kucharz, Rational maps in real algebraic geometry, *Adv. Geom.* **9** (2009), 517D539.
- [25] W. Kucharz, Approximation by continuous rational maps into spheres, *J. Eur. Math. Soc.* **16** (2014), 1555D1569
- [26] T.-C. Kuo, On classification of real singularities, *Invent. math.* **82** (1985), 257–262.
- [27] K. Kurdyka, Ensembles semi-algébriques symétriques par arcs, *Math. Ann.* **282** (1988), 445–462
- [28] K. Kurdyka, Points réguliers d’un sous-analytique, *Annales de l’institut Fourier* **38**, 1, (1988), 133–156
- [29] K. Kurdyka, Injective endomorphisms of real algebraic sets are surjective. *Math. Ann.* **313** (1999), no. 1, 69–82.
- [30] K. Kurdyka, A. Parusiński, *Arc-symmetric Sets and Arc-analytic Mappings*, dans "Arc Spaces and Additive Invariants in Real Algebraic Geometry", Proceedings of Winter School "Real algebraic and Analytic Geometry and Motivic Integration", Aussois 2003, eds. M. Coste, K. Kurdyka, A. Parusiński, Panoramas et Synthèses **24**, 2008, 33–68
- [31] D. T. Lê, B. Teissier, Limites d’espaces tangents en géométrie analytique, *Comment. Math. Helvetic* **63** (1988), 540–578
- [32] T. Limoges, F. Priziac, Cohomology and products of real weight filtrations, *Annales de l’institut Fourier* to appear
- [33] S. Łojasiewicz, Ensembles semi-analytiques, Preprint IHES (1965), see also <https://perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf>.
- [34] I. Luengo, An example concerning a question of Zariski. *Bull. Soc. Math. France* **113** (1985), no. 4, 379–386.
- [35] C. McCrory, A. Parusiński, Virtual Betti numbers of real algebraic varieties, *C.R.A.S. Ser. I* **336** (2003), 763–768.
- [36] C. McCrory, A. Parusiński, The weight filtration for real algebraic varieties, *MSRI Publications*, **18** "Topology of Stratified Spaces" (2011), 121–160
- [37] C. McCrory, A. Parusiński, *The weight filtration for real algebraic varieties II: Classical homology*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. (volume II en l’honneur de H. Hironaka), (2014) **108**, 63–94

- [38] J. Milnor, Singular points of complex hypersurfaces. Annals of Mathematics Studies, No. 61 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968
- [39] T. Mostowski, Topological equivalence between analytic and algebraic sets, *Bull. Polish Acad. Sci. Math.*, **32**, (1984), no. 7–8,
- [40] T. Mostowski, Lipschitz Equisingularity, *Dissertationes Math.* **243** (1985).
- [41] C. Murolo, D. J. A. Trotman, Semi-differentiable stratified morphisms. *C. R. Acad. Sci. Paris Sér. I Math.* **329** (1999), no. 2, 147–152
- [42] C. Murolo, D. J. A. Trotman, Horizontally- $C^1$  controlled stratified maps and Thom's first isotopy theorem. *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), no. 8, 707–712
- [43] C. Murolo, D. J. A. Trotman, Semidifférentiabilité et version lisse de la conjecture de fibration de Whitney. Singularity theory and its applications, 271–309, *Adv. Stud. Pure Math.*, **43**, Math. Soc. Japan, Tokyo, 2006.
- [44] R. Narasimhan, "Introduction to the theory of analytic spaces", Lect. notes in math., 25, Springer (1966)
- [45] A. Parusiński, Algebraic equisingularity and Whitney's conditions. *Bull. Polish Acad. Sci. Math.* **33** (1985), no. 5-6, 247–253.
- [46] A. Parusiński, *Lipschitz stratification*, in "Global Analysis in Modern Mathematics", K. Uhlenbeck, ed., Proceedings of a symposium in Honor of Richard Palais' Sixties Birthday, Publish or Perish, Houston, 1993, pp. 73–91
- [47] A. Parusiński, Limits of tangent spaces to fibres and the  $w_f$  condition, *Duke Math. Jour.* **72**, 1 (1993), 99–109
- [48] A. Parusiński, Subanalytic functions, *Trans. Amer. Math. Soc.* **344**, 2 (1994), 583–595.
- [49] A. Parusiński, Topology of injective endomorphisms of real algebraic sets, *Math. Annalen*, **328** (2004), 353–372.
- [50] A. Parusiński, G. Rond, The Abhyankar-Jung Theorem, *Journal of Algebra* 365 (2012) 29-41
- [51] W. Pawłucki, Le théorème de Puiseux pour une application sous-analytique *Bull. Polish Acad. Sci. Math.* **32** (1984), no. 9–10, 555–560
- [52] J.-P. Speder, Equisingularité et Conditions de Whitney, *American Journal of Mathematics*, Vol. **97**, No. 3 (1975), pp. 571–588
- [53] J.-P. Speder, Equisingularité et Conditions de Whitney, *Thèse Université de Nice*, 1976
- [54] A. N. Varchenko, Theorems on the topological equisingularity of families of algebraic varieties and families of polynomial mappings. *Math. USSR Izvestija*, Vol. **6** (1972), 949–1008.
- [55] A. N. Varchenko, The relation between topological and algebro-geometric equisingularities according to Zariski. *Funkcional. Anal. Appl.* **7** (1973), 87–90.
- [56] A. N. Varchenko, Algebro-geometrical equisingularity and local topological classification of smooth mappings. Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 1, pp. 427–431. Canad. Math. Congress, Montreal, Que., 1975.
- [57] J.-L. Verdier, Stratifications de Whitney et théorème de Bertini-Sard, *Inventiones Math.* **36** (1976), 295–312.
- [58] O. U. Villamayor, On equiresolution and a question of Zariski. *Acta Math.* **185** (2000), no. 1, 123–159.
- [59] C. Voisin, Hodge theory and complex algebraic geometry. Cambridge Studies in Advanced Mathematics, 77. Cambridge University Press, Cambridge, 2003
- [60] A. H. Wallace, Algebraic approximation of curves, *Canadian J. Math.* 10 (1958), 242–278.
- [61] H. Whitney, Tangents to an analytic variety. *Ann. of Math.* (2) **81** 1965 496–549.
- [62] H. Whitney, Local properties of analytic varieties. 1965 Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse) pp. 205–244 Princeton Univ. Press, Princeton, N. J.
- [63] O. Zariski, Studies in equisingularity. I. Equivalent singularities of plane algebroid curves. *Amer. J. Math.* **87** (1965) 507–536; Oscar Zariski: Collected Papers, Volume IV, MIT Press, 31–60

- [64] O. Zariski, Studies in equisingularity. II. Equisingularity in codimension 1 (and characteristic zero). *Amer. J. Math.* **87** (1965) 972–1006; Oscar Zariski: Collected Papers, Volume IV, MIT Press, 61–95
- [65] O. Zariski, Some open questions in the theory of singularities, *Bull. Amer. Math. Soc.* **77** (1971) 481–491; Oscar Zariski: Collected Papers, Volume IV, MIT Press, 238–248
- [66] O. Zariski, On Equimultiple Subvarieties of Albroid Hypersurfaces, *Proc. Nat. Acad. Sci. USA* **72**, No. 4, (1975) 1425–1426; Oscar Zariski: Collected Papers, Volume IV, MIT Press, 525–526
- [67] O. Zariski, Foundations of a General Theory of Equisingularity on  $r$ -Dimensional Albroid and Algebraic Varieties, of Embedding Dimension  $r + 1$ , *Amer. J. Math.* **101**, 2, (1979) 453–514; Oscar Zariski: Collected Papers, Volume IV, MIT Press, 573–635

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